Adjustment and Gross Errors Detection of Free Triangulation Geodetic Network Using Minimum-Norm Least-Squares Inverses and Data Snooping

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Abstract: We utilise minimum-norm least-squares based on the indirect observations methods to adjust our 2-dimensional triangulation network. The main objective of this paper is to optimally adjust the approximate coordinates of the nodes (points) of the given network. The network observations (11 measured distances and 17 angles) have been adjusted by being combined in linear system of equations in terms of free-network adjustment procedure to rigorously adjust the approximate coordinates over the network points. We obtained better converged values by applying an iterative procedure, the minimum corrections for the free-network coordinates are obtained after a number of five iterations. The data snooping procedure has been used to test the reliability and precision of the network observations. The T-Test criterion is then applied for gross error detection, five angles and two lines are suspected to include gross errors at a critical value of 1.98.

Keywords: Minimum-norm least-squares, triangulation network, iterative procedure, free-network, data snooping procedure, T-Test criterion, critical value.

1. INTRODUCTION

In surveying profession, the unknown quantities such as coordinates, bearings and angles between the geodetic lines are derived from the field observations [1, 2]. There are two types of field observations, namely, the direct and indirect observations. The direct observations are explicitly applied to measure various quantities e.g. lengths, angles and bearings using specific instruments such as tape, Total Station or Theodolite and compass. On the other hand, the indirect observations are made to compute a quantity which can not be obtained directly, for instance angles and distances that connecting points are used to derive the coordinates of the unknown points based on the mathematical relationships which combine these quantities together [2]. The quantities derived from the indirect observations are always liable to errors that propagate from the direct observations. Therefore, various adjustment techniques were introduced to reduce these errors e.g. Bowditch's and Transit Rules for the adjustment of the traverse closing error, where the closing error arises from the combination of the observations and coordinates errors [3, p. 105].

It is believed that there are no entirely correct measurements, meaning that each measurement no matter how precise it is, likely to contain errors. These errors happen due to various reasons e.g. human mistakes (gross errors), instruments and weather (systematic errors) [4]. One more type of the generalized errors are the random errors which are found in all surveying measurements and remain even after the elimination of the gross and systematic errors because it is not easy to detect or eliminate them. These errors are modeled by the probability laws, so they can be adjusted easily. The adjustment for the magnitude of the observations in different stations represents a frequent problem in geodesy and surveying due to the situation and the way of conducting the observations [5].

In literature, many authors have contributed in the network optimisation. A general review of network design based on cost and design orders were conducted by [6], other studies in designing the second and third order networks were also conducted by [7] and [8], respectively [9]. Employed the generalized matrix inverses to design zero order network and to dene a datum for geodetic network. More related studies in network design and optimisation are found in [10, 11]. Considered a method of a free-network adjustment which minimises the sum of the squares of the weighted errors matrix and the euclidean norm of the vector of unknowns and the covariance matrix. In addition, the geodetic free-network was reviewed by [12]. Moreover, [13] used free-network adjustment techniques to obtain a minimum trace of the variance-covariance matrix of the parameters [14].

The optimisation of the geodetic networks is conducted by considering the precision, reliability and low cost aspects. A systematic and comprehensive

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analysis of the geodetic network design and optimisation is compiled in [5, 15], have used three models based on these aspects to optimise the design of the geodetic network. They found that among other models the reliability model to be better for geodetic network optimisation.

In this study, we employ least-squares adjustment of a non-constrained 2-dimensional triangulation network (free-network) using indirect observations method [16, 17]. Specifically, the observed angles, distances and approximate coordinates are precisely optimised in a rigorous least-squares way. After conducting a number of iterations, we yield the optimal coordinates for each point with approximate coordinates (float points). The iteration procedure is considered to obtain an optimal adjustment for our network. The corrections of the triangulation float points in the current network have been significantly minimised and refined at each new iteration until the convergence between the iterated and optimal coordinates becomes a minimum.

The organisation of this paper is based on six sections, following to this introduction, the free-network methodology is addressed in Section 2, the modelling of the observations is explained in Section 3). The Data Snooping criterion and post-adjustment technique applied in this study are found in Section 4. The numerical results are investigated and analysed in section 5. Concluding remarks on the obtained results are drawn in section 6.

2. FREE-NETWORK ADJUSTMENT

The geodetic network is called to be a free network when it lacks the essential information such as position, orientation and scale of the network and the datum parameters of the coordinate reference system [14, 17]. The measurements of the baselines are based on observational process either during the triangulation or in the trilateration [14]. The adjustment of the free networks is highly important as it consists of several practical applications. Despite the missing of the essential information, it is still possible to solve the free network in terms of least-squares and obtaining the adjusted coordinates and analysing the least-squares residuals [18].

The free geodetic network is solved by the normal least-squares formed by a number of observation equations [17]

$$L = \underset{n \cdot m}{A} X \qquad (1)$$

where L denotes the vector of the reduced observations, A is the $n \times m$ design matrix which and X is the unknowns vector the evaluation of the linear least-squares is normally solved according to the following relation

$$\min_{X \in R^n} \left\| AX - b \right\| \qquad (2)$$

where $X \in R^{m \times n}$, $b \in R^m$

In the normal cases, Equation 1 can be solved properly while the design matrix A is well-posed and of full rank one. On the other hand, the matrix manipulation by means of leas-squares does not go well when the design matrix A is ill-posed and suffers from a rank-deficiency problem. In such case, the solution of Equation 1 requires using one of the regularisation procedures e.g. singular value decomposition (SVD), Tikhonov regularisation or rank-revealing QR factorization (RRQR), more information can be found in [19-21].

However, the free network adjustment method has problems associated with the singularity matrix A as a result of the linear dependence of its columns, respectively. This is due to the loss of the rank of the configuration matrix, and it is also referred to the network defect. Necessary parameters are needed to eliminate the degrees of freedom of geodetic network in the processing area.

The design matrix A in Equation 1 will be singular due to the rank defect d

$$d = m - r \qquad (3)$$

$$r\left(\underset{n \cdot m}{A}\right) = r < m < n \qquad (4)$$

The minimum-norm least-squares solution is explained clearly in [17]. The inconsistent linear equation system is regularised by considering the vector of the unknowns ϵ as follows;

$$\underset{n \cdot m}{A} \underset{m \cdot 1}{X} = \underset{n \cdot 1}{L} - \underset{n \cdot 1}{\epsilon} \qquad \left[r\left(A\right) \le m < n \right] \qquad (5)$$

The solution of the above equation (5) is not unique because X and ϵ are the unknown and the residuals vectors which fulfill the least-squares condition

$$\epsilon^T P \epsilon = \min \qquad (6)$$

where P is the square matrix that denotes the associated weight of the network observations (lines and angles)

The derivation of the least-squares solution, we start from the general solution of the Equation 5. Considering the condition shown in Equation 4 where the system of equations is inconsistent and over-determined with rank defect. The least-squares solution based on the following form of equation will not be unique:

$$\left(L - AX\right)^T P\left(L - AX\right) = \min \tag{7}$$

The least-squares general solution is obtained in the following form [17, eq. 4.110, P. 150];

$$\begin{aligned} \widetilde{X} &= A^{-1}L + \left(I - A^{-1}A\right)V \\ &= \widetilde{A}^{-}L \end{aligned} \tag{8}$$

where $\widetilde{A}^{-}$ is the generalised inverse of matrix A and V is an arbitrary vector;

$$\widetilde{A}^{-} = \left(A^T PA + D^T D\right)^{-1} A^T P \tag{9}$$

D is defined such that;

$$\begin{aligned} &AD^T = 0 \\ &\left|DD^T\right| \neq 0 \\ &\left|A^T PA + D^T D\right| \neq 0 \end{aligned} \tag{10}$$

The structure of matrix D is based on the structure of the design matrix A . Therefore, the elements of D vary between different types of free networks e.g. 1-D levelling networks, 2-D triangulation networks with distances, angles or directions. The structure of D matrix in 2-D triangulation networks has three formats with respect to the types of the observations (e.g. distances, angles, directions) as follows:

2-D Triangulation networks with distance observations $\left(d = 3\right)$;

$$\underset{3\cdot m}{D} = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 1 \\ -N_1^0 & E_1^0 & -N_2^0 & E_2^0 & \dots & -N_k^0 & E_k^0 \end{bmatrix} \tag{11}$$

where m is the number of unknown parameters, $k = m / 2$ is the number of unknown points, $\left(E_i^0, N_i^0\right)$ are the approximate coordinates of network.

2-D Triangulation networks with angles observations $\left(d = 4\right)$:

$$\underset{4\cdot m}{D} = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 1 \\ -N_1^0 & E_1^0 & -N_2^0 & E_2^0 & \dots & -N_k^0 & E_k^0 \\ E_1^0 & N_1^0 & E_2^0 & N_2^0 & \dots & E_k^0 & N_k^0 \end{bmatrix} \tag{12}$$

where m is the number of unknown parameters, $k = m / 2$ is the number of unknown points, $\left(E_i^0, N_i^0\right)$ are the approximate coordinates of network.

2-D Triangulation networks with direction observations $\left(d = 4\right)$.

The variance-covariance matrix of $\hat{X}$ is obtained as follows

$$\begin{aligned} C_{\hat{x}\hat{x}} &= \left(A^T PA + D^T D\right)^{-1} A^T P \sigma_0^2 P^{-1} \cdot \left[\left(A^T PA + D^T D\right) A^T P\right]^T \\ &= \sigma_0^2 \left(\overline{N}^{-1} - \overline{N}^{-1} D^T D \overline{N}^{-1}\right) \end{aligned} \tag{13}$$

3. OBSERVATIONAL MODELLING

The combined triangulation and trilateration network in Figure **1** is based on a total of 28 observations, 17 angles and 11 distances. All these observations are connected to 6 benchmarks to form the free network.

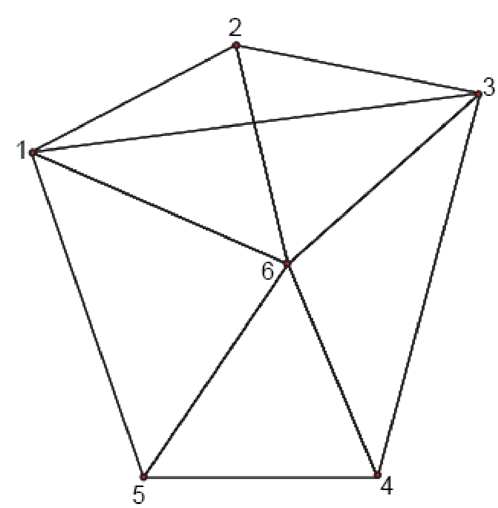


Figure 1: A newly-measured triangulation network with unadjusted (17) angles and (11) distances.

The network observations have been measured by a total station instrument, the approximate coordinates are electronically calculated based on the functionality of the instrument (using resection and intersection methods). The combination of the triangulation and trilateration is a robust network that used to create horizontal control points based on such traditional methods [2, section 9.7, p 501].

3.1. Angles Condition Equations

As shown in Figure **1**, we need to create the condition equations for both angles and distances measured in the network. The condition equations of angles are written in terms of azimuths, because the horizontal angle ijk can be derived by subtracting the azimuth of the line ij from the azimuth of line ik. As the azimuth represents the horizontal angle measured from a north meridian clockwise to a line, one may be cautious when computing the azimuth in Equation 14 and make sure that it is located in the right quadrant. Otherwise, it will subsequently lead to wrong results in the adjustment process due to the geometrical errors in the network [5].

The azimuth condition equations for α_{ij} and α_{ik} are written as follows:

$$\alpha_{ij} = \arctan\frac{E_j^0 - E_i^0}{N_j^0 - N_i^0} \text{ and } \alpha_{ik} = \arctan\frac{E_k^0 - E_i^0}{N_k^0 - N_i^0} \quad (14)$$

where E_i^0, E_j^0, E_k^0 and N_i^0, N_j^0, N_k^0 are the coordinates at the ends of the lines ℓ_{ij} and ℓ_{jk}, respectively.

The condition equation of angle θ_{ijk} is written as:

$$\begin{aligned}\theta_{ijk} + v_{ijk} &= \alpha_k - \alpha_i \\ &= \arctan\frac{E_k^0 - E_j^0}{N_k^0 - N_j^0} - \arctan\frac{E_i^0 - E_j^0}{N_i^0 - N_j^0}\end{aligned} \quad (15)$$

where v_{ijk} denote the residuals of the angles. The linearisation of the Equation 15 is obtained by the followings:

putting;

$$u_k = \frac{E_k^0 - E_j^0}{N_k^0 - N_j^0}, u_i = \frac{E_i^0 - E_j^0}{N_i^0 - N_j^0} \dots u_n = \frac{E_n^0 - E_j^0}{N_n^0 - N_j^0} \quad (16)$$

when executing the partial derivative of equation 16, we get;

$$\frac{\partial}{\partial E}\arctan u = \frac{1}{1+u^2}\cdot\frac{\partial u}{\partial E} \quad (17)$$

then the linearised form of Equation 15 is given as;

$$\left(\theta_{ijk} - \bar{\theta}_{ijk}\right) - v_{ijk} = a.\delta E_i + b.\delta N_i + c.\delta E_j + d.\delta N_j + e.\delta E_k + f.\delta N_k \quad (18)$$

Using the partial derivative with respect to the approximate coordinates $\left(E_i^0, N_i^0\right)$, $\left(E_j^0, N_j^0\right)$ and $\left(E_k^0, N_k^0\right)$, the evaluation of the coefficients in Equation (18) is computed by the following equations:

$$\begin{aligned}a &= \frac{\partial\theta_{ijk}}{\partial E_i} = \frac{\sin\alpha_{ik}}{\tilde{\ell}_{ik}} - \frac{\sin\alpha_{ij}}{\tilde{\ell}_{ik}} \\ b &= \frac{\partial\theta_{ijk}}{\partial N_i} = -\frac{\cos\alpha_{ik}}{\tilde{\ell}_{ik}} + \frac{\cos\alpha_{ij}}{\tilde{\ell}_{ik}} \\ c &= \frac{\partial\theta_{ijk}}{\partial E_j} = \frac{\sin\alpha_{ij}}{\tilde{\ell}_{ij}} \\ d &= \frac{\partial\theta_{ijk}}{\partial N_j} = -\frac{\cos\alpha_{ij}}{\tilde{\ell}_{ik}} \\ e &= \frac{\partial\theta_{ijk}}{\partial E_k} = -\frac{\sin\alpha_{ik}}{\tilde{\ell}_{ik}} \\ f &= \frac{\partial\theta_{ijk}}{\partial N_k} = \frac{\cos\alpha_{ik}}{\tilde{\ell}_{ik}}\end{aligned} \quad (19)$$

The following parameters $\bar{\theta}_{ijk}, \tilde{\ell}_{ij}, \tilde{\ell}_{ik}$ are computed by the approximate coordinates by the following equations;

$$\begin{aligned}\bar{\theta}_{ijk} &= \alpha_{ik} - \alpha_{ij} \\ \tilde{l}_{ij} &= \left[\left(E_i^0 - E_j^0\right)^2 + \left(N_i^0 - N_j^0\right)^2\right]^{1/2} \textit{ and} \\ \tilde{l}_{ik} &= \left[\left(E_i^0 - E_k^0\right)^2 + \left(N_i^0 - N_k^0\right)^2\right]^{1/2}\end{aligned} \quad (20)$$

3.2. Distances Condition Equations

The distance condition shows that the distance ℓ_{ij} between two points i and j as a function F_{ij} of the coordinates $\left(E_i^0, N_i^0\right)$ and $\left(E_j^0, N_j^0\right)$ as represented in the following functional equation:

$$\ell_{ij} + v_{ij} + F_{ij}\left(E_i^0, N_i^0, E_j^0, N_j^0\right) = 0 \quad (21)$$

$$l_{ij} + v_{ij} - \left[\left(E_i^0 - E_j^0\right)^2 + \left(N_i^0 - N_j^0\right)^2\right]^{1/2} = 0 \quad (22)$$

The linearisation of Equation (22) is obtained as follows;

$$\left(\ell_{ij} - \tilde{\ell}_{ij}\right) - v_{ij} = a\cdot\delta E_i + b\cdot\delta N_i + c\cdot\delta E_j + d\cdot\delta N_j \quad (23)$$

Similarly, using the partial derivative with respect to the approximate coordinates $\left(E_i^0, N_i^0\right)$ and $\left(E_j^0, N_j^0\right)$, the evaluation of the coefficients in Equation (23) is computed by the following equations:

$$
\begin{aligned}
a &= \frac{\partial \ell_{ij}}{\partial E_i} = -\frac{E_j^0 - E_i^0}{\tilde{\ell}_{ik}} \\
b &= \frac{\partial \ell_{ij}}{\partial N_i} = -\frac{N_j^0 - N_i^0}{\tilde{\ell}_{ik}} \\
c &= \frac{\partial \ell_{ij}}{\partial E_j} = \frac{E_j^0 - E_i^0}{\tilde{\ell}_{ik}} \\
d &= \frac{\partial \ell_{ij}}{\partial N_j} = \frac{N_j^0 - N_i^0}{\tilde{\ell}_{ik}}
\end{aligned}
\tag{24}
$$

6. SOLUTION OF FREE-NETWORK LEAST-SQUARES MODEL

The creation of the design matrix A in Equation 8 is based on the coefficients a,b,c,d,e,f as shown in equations 19 and 24. The dimension of the design matrix is determined by computing the number of the coordinates points in the network which will represent the number of columns. While the rows are represented by the combination of the angles and distances condition equations, respectively. For instance, the current network of this study contains a number of six benchmarks, 17 angles and 11 distances. Since each single benchmark consists of a pair of coordinates, this means that the number of the columns in the design matrix for any network are equal to $\left(2\times n\right)$ where n denotes the number of the benchmarks. The number of the rows is computed by the adding the total number of the linearised conditions equations for angles and distances, in this study they are found to be 28 rows. Hence, the exact dimension of the the design matrix A based on the network observations and their final linearised condition equations is $\underset{12\times 28}{A}$.

The rest of points which are not involved are considered as zeros in the corresponding columns. The following example indicates the criterion of building up the design matrix A based on the aforementioned information.

Referring to Figure **1**, the organisation of the coefficients starts first with all linearised angles equations (rows $1 \rightarrow 17$), then follows with distances (rows $18 \rightarrow 28$). For instance, we can take the first two angles θ_{213}, θ_{316} and the last linearised distance ℓ_{56} equation in the last row of the A matrix as follows:

$$
A = \begin{bmatrix}
\delta E_1^0 & \delta N_1^0 & \delta E_2^0 & \delta N_2^0 & \delta E_3^0 & \delta N_3^0 & \delta E_4^0 & \delta N_4^0 & \delta E_5^0 & \delta N_5^0 & \delta E_6^0 & \delta N_6^0 \\
b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
b_{21} & b_{22} & 0 & 0 & b_{25} & b_{26} & 0 & 0 & 0 & 0 & b_{211} & b_{212} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{289} & b_{2810} & b_{2811} & b_{2812}
\end{bmatrix}
\tag{25}
$$

The approximate coordinates in the last row of matrix D in Equation (11), are preferably smoothed by subtracting their mean coordinates $\left(\overline{E},\overline{N}\right)$, so D can be written as follows:

$$
\underset{3\cdot 6}{D} = \begin{bmatrix}
1 & 0 & 1 & 0 & \ldots & 1 & 0 \\
0 & 1 & 0 & 1 & \ldots & 0 & 1 \\
-\left(N_1^0 - \overline{N}\right) & \left(E_1^0 - \overline{E}\right) & -\left(N_2^0 - \overline{N}\right) & \left(E_2^0 - \overline{E}\right) & \ldots & -\left(N_6^0 - \overline{N}\right) & \left(E_6^0 - \overline{E}\right)
\end{bmatrix}
\tag{26}
$$

the corrections of the approximate coordinates are obtained in the unknown vector $\widetilde{X}$ as follows;

$$
\widetilde{X} = \left[\widetilde{\delta E_1} \quad \widetilde{\delta N_1} \quad \widetilde{\delta E_2} \quad \widetilde{\delta N_2} \quad \ldots \quad \widetilde{\delta E_6} \quad \widetilde{\delta N_6}\right]^T
\tag{27}
$$

By obtaining the coordinate corrections vector $\widetilde{X}$ from Equation (8), the approximate coordinates $\left(E_i^0, N_i^0\right)$ are corrected which also means that the network is now adjusted. Since the corrections $\left(\widetilde{\delta E_i}, \widetilde{\delta N_i}\right)$ are added to the approximate coordinates, there will be a slight shift due to the change in the coordinates:

$$
\begin{bmatrix} \hat{E}_i \\ \hat{N}_i \end{bmatrix} = \begin{bmatrix} E_i^0 \\ N_i^0 \end{bmatrix} + \begin{bmatrix} \widetilde{\delta E_i} \\ \widetilde{\delta N_i} \end{bmatrix}
\tag{28}
$$

The radial distances $\overrightarrow{OP_i}$ and $\overrightarrow{O\widetilde{P}_i}$ from the origin coordinate centre $O\left(0,0\right)$ to P_i and $\widetilde{P}_i$ are:

$$
r_i^0 = \left[\left(E_i^0\right)^2 + \left(N_i^0\right)^2\right]^{1/2} \textit{ and } \tilde{r}_i = \left[\left(\tilde{E}_i\right)^2 + \left(\tilde{N}_i\right)^2\right]^{1/2}
\tag{29}
$$

and their corresponding azimuths are given by:

$$
\alpha_i^0 = \arctan\left(\frac{N_i^0}{E_i^0}\right) \text{ and} \tilde{\alpha}_i^0 = \arctan\left(\frac{\widetilde{N_i}^0}{\widetilde{E_i}^0}\right)
\tag{30}
$$

The total area $\hat{A}$ swept by the radial distances $\overrightarrow{OP_i}$ and $\overrightarrow{O\widetilde{P}_i}$ due to the changes indicated in Equation (28) is given by:

$$
\hat{A} = \frac{1}{2}\sum_{i=1}^{k}\left(-\widetilde{N_i}^0 \cdot \widetilde{\delta E_i} + \widetilde{E_i}^0 \cdot \widetilde{\delta N_i}\right)
\tag{31}
$$

4. POST-ADJUSTMENT TECHNIQUE

4.1. Data Snooping

Data snooping is a procedure used for the detecting and localising the gross errors in the adjusted observations [22-24]. It is a justified statistical derivation that is conducted to make a decision on the tested observations after least-squares adjustment [25]. The gross error is experimentally located by performing a one-by-one comparison of the least-squares residuals versus their standard errors, this comparison can similarly be repeated.

The functional and statistical models for the null hypothesis H_0 stands for an assumption that considers the observations as gross-error-free, they are presented as follows in [17];

$$H_0 : \begin{Bmatrix} \underset{n\times 1}{L} - \underset{n\times 1}{\in} = \underset{n\times m}{A} \; \underset{m\times 1}{X} \\ E\{\varepsilon\} = 0 \\ E\{\varepsilon\varepsilon^T\} = \sigma_0^2 Q = \sigma_0^2 P^{-1} \end{Bmatrix} \tag{32}$$

where P and Q are the weight and co-factor matrice of ϵ .

The variance-covariance matrix $C_{\hat{\varepsilon}\hat{\varepsilon}}$ and co-factor matrix $Q_{\hat{\varepsilon}\hat{\varepsilon}}$ by the following equation:

$$C_{\hat{\varepsilon}\hat{\varepsilon}} = \sigma_0^2 \cdot Q_{\hat{\varepsilon}\hat{\varepsilon}} \tag{33}$$

$$Q_{\hat{\varepsilon}\hat{\varepsilon}} = P^{-1} - A\left(A^T P A\right)^{-1} A^T \tag{34}$$

The substitutional hypothesis H_1 assumes that one of the observations contains gross error Δi :

$$H_1 : \begin{Bmatrix} \underset{n\times 1}{L} - \underset{n\times 1}{\in} = \underset{n\times m}{A} \; \underset{m\times 1}{X} + \underset{n\times 1}{ei} \cdot \Delta i \\ E\{\varepsilon\} = 0 \\ E\{\varepsilon\varepsilon^T\} = \sigma_0^2 Q = \sigma_0^2 P^{-1} \end{Bmatrix} \tag{35}$$

where e_i denotes a vector of zero elements except the i-th element which equals 1.

4.2. Observations Quality

The precision of the adjusted observation is still not sufficient, because the adjusted observation must be precise and reliable too. The reliability of any observation stands for its proximity to the true value, better reliability can be obtained by repetitions and iterations. The network quality can be described by precision and reliability in terms of variance-covariance matrix and the adjusted quantities [17]. There are two types of reliability, namely internal and external. The internal reliability is characterised by the minimal detectable error Δ_i (see Equation 35) which shows the amount of the smallest possible observation error which is detected by the statistical test. The redundancy number can also be used for the internal reliability. A large redundancy number indicates proximity to the true value which means a strong reliability. Conversely, a smaller the redundancy number denotes poor reliability and when the redundancy number equals zero it means that the observation is not checked at all.

The internal reliability is computed by local redundancy κ_i of the observation ℓ_i and its standard error as follows;

$$\begin{aligned} \hat{\Delta} i &= \frac{\sigma_0 \cdot \sqrt{q_{ii}}}{p_{ii} \cdot q_{ii}} \cdot u_i \\ &= \frac{\sigma_0 / \sqrt{q_{ii}}}{\sqrt{p_{ii} \cdot q_{ii}}} = \frac{\sigma_i}{\sqrt{k_i}} \cdot u_i \end{aligned} \tag{36}$$

where q_{ii} denotes the $i-th$ diagonal element in the cofactore matrix $Q_{\hat{\varepsilon}\hat{\varepsilon}}$, p_{ii} denotes the $i-th$ diagonal element in the weight matrix P as shown in Equation 34. The standard error for any observation σ_i is obtained by:

$$\sigma_i = \frac{\sigma_0}{\sqrt{p_{ii}}} \tag{37}$$

The minimum detectable gross error is given by:

$$\theta_i = \frac{\sigma_i}{\sqrt{\kappa_i}} \cdot \delta_0\left(\alpha, \beta\right) \tag{38}$$

where $\delta_0\left(\alpha,\beta\right)$ is the non-central shift in the standard normal distribution and it is given by:

$$\delta_0\left(\alpha, \beta\right) = c_{\frac{1}{2}\alpha} + c_{1-\beta} \tag{39}$$

where $c_{\frac{1}{2}\alpha}$ is the standard normal distribution at the risk level α , and $1-\beta$ is the power of the statistical test.

The effect of the undetected gross error on the observation is called the external reliability of the observation. It is a measure to compute the impact of a possible error on the observations or constrained stations on the adjusted ones. So the minimum detectable effect on Δ_i on the adjusted observation $\overline{\ell}_i$ is given by;

$$\varphi_i = \left(1 - \kappa_i\right) \cdot \frac{\sigma_i}{\kappa_i} \delta_0\left(\alpha, \beta\right) = \left(1 - \kappa_i\right) \cdot \theta_i \quad (40)$$

Alternatively, the external reliability is evaluated by computing the bias to noise ratio ϑ_i for each observation, which reflects the influence of the possible gross error in the observation.

$$\left|\frac{\nabla_{E_i}}{\sigma_{E_i}}\right| \leq \vartheta_i \quad (41)$$

5. NUMERICAL ANALYSIS

The adjustment procedure followed in this study is presented in the flow chart illustrated in Figure **2**, the steps we followed are the least-squares adjustment using the minimum-norm method which is used for the geodetic free network. The data snooping technique is employed to investigate the quality of the network observations. The quality of the geodetic network can be evaluated in terms of precision and reliability which are numerically represented by certain parameters as explained in Equations 38, 40 and 41.

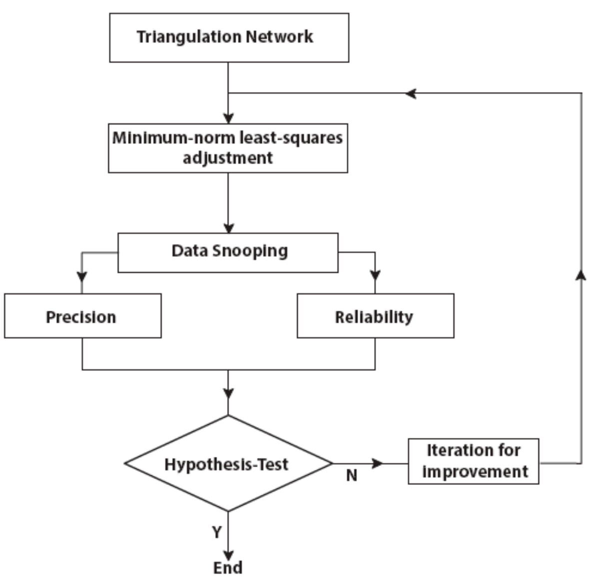
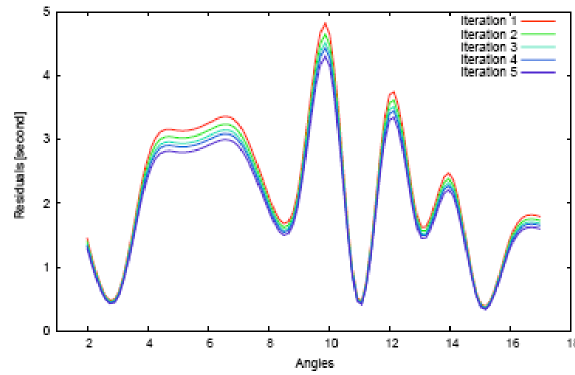


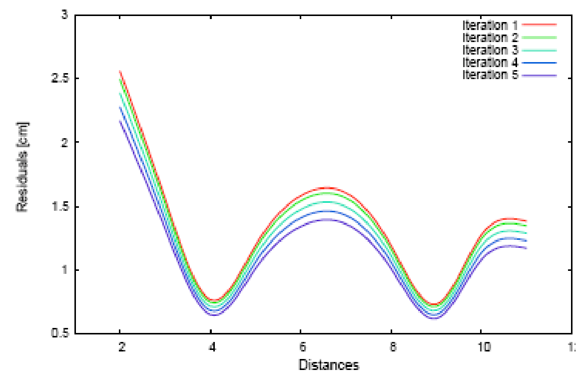
Figure 2: Adjustment procedure including test criterion and improvement iteration.

5.1. Minimum-Norm Least-Squares Adjustment

In this study, five iterations have been performed for the sake of improvement. Figure **3** shows that the residuals of the observations v_i (angles and distances) are significantly reduced from iteration 1 to 5. It is also clear that the residuals are very close in some angle observations over all iterations, while a noticeable convergent can also be seen at the rest of the observations (cf. Figure **3a**). In the same manner, the residuals of the distance in Figure **3b** are also reduced with respect to the iterations.



(**a**) Angles



(**b**) Distance

Figure 3: Residuals in angles and distances after five iterations.

The impact of the iterations on the radial distances and their azimuths is computed and illustrated in Tables **1** and **2**. This can also be confirmed by calculating the total area wept by the radial distances after the final adjustment which is appearing insignificant, this means that the shift of the network coordinates is negligible because it is assumed that when $\hat{A} = 0$, which means that the network has zero rotation during the free-network adjustment.

5.2. Mathematical and Stochastic Models

The data snooping technique is utilised in the second step of the adjustment procedure. The numerical evaluations of the precision and reliability of the angles and distances are illustrated in Tables **3** and **4**, respectively. The internal reliability of the angles θ_i in Table **3** are based on the minimum detectable gross error and are expressed in terms of seconds. It is found that the minimum detectable gross error θ_i varies in each angle, it reaches up to 21 arc-second in $\angle 3,6,1$

Table 1: The Iterations Impact on the Radial Distances r_i^0 and $\tilde{r}_i$

	I		II	
	r_i^0	$\tilde{r}_i$	r_i^0	$\tilde{r}_i$
1	1778379.237	1778376.546	1778379.264	1778379.237
2	1779505.867	1779505.917	1779505.917	1779505.867
3	1779158.691	1779133.693	1779158.941	1779158.691
4	1777599.453	1777664.695	1777598.80	1777599.453
5	1776594.230	1776584.288	1776594.329	1776594.229
6	1778211.039	1778188.381	1778211.265	1778211.039
	III		**IV**	
	r_i^0	$\tilde{r}_i$	r_i^0	$\tilde{r}_i$
1	1778379.270	1778377.615	1778379.258	1778377.615
2	1779505.951	1779501.780	1779505.899	1779501.830
3	1779159.244	1779131.627	1779159.231	1779131.627
4	1777598.142	1777663.659	1777598.315	1777663.124
5	1776594.420	1776584.880	1776594.391	1776585.807
6	1778211.489	1778188.951	1778211.812	1778188.752
	V			
	r_i^0	$\tilde{r}_i$		
	1778379.301	1778377.721		
	1779505.913	1779501.871		
	1779159.452	1779131.674		
	1777598.215	1777663.597		
	1776594.753	1776584.778		
	1778211.489	1778188.790		

Table 2: The Iterations Impact on the Azimuths of the Radial Distances r_i^0 and $\tilde{r}_i$

	I		II	
	α_i^0	$\tilde{\alpha}_i^0$	α_i^0	$\tilde{\alpha}_i^0$
1	75° 06′ 29.465″	75° 06′ 27.935″	75° 06′ 29.480″	75° 06′ 29.465″
2	75° 03′ 39.108″	75° 03′ 39.330″	75° 03′ 39.106″	75° 03′ 39.108″
3	74° 59′ 17.616″	74° 59′ 14.491″	74° 59′ 17.647″	74 ∘ 59′ 17.616″
4	75° 00′ 21.702″	75° 06′ 26.304″	75° 00′ 21.665″	75° 00′ 21.702″
5	75° 04′ 59.862″	75° 05′ 0.448″	75° 04′ 59.028″	75° 04′ 59.862″
6	75° 02′ 59.020″	75° 02′ 58.274″	75° 02′ 59.028″	75° 02′ 59.020″
	III		**IV**	
	α_i^0	$\tilde{\alpha}_i^0$	α_i^0	$\tilde{\alpha}_i^0$
1	75° 06′ 29.497″	75° 06′ 27.916″	75° 06′ 29.512″	75° 06′ 27.890″
2	75° 03′ 39.093″	75° 03′ 39.849″	75° 03′ 38.936″	75° 03′ 39.607″
3	74° 59′ 17.683″	74° 59′ 17.334″	74° 59′ 17.523″	74° 59′ 17.624″
4	75° 00′ 21.622″	75° 06′ 26.644″	75° 00′ 21.622″	75° 00′ 26.644″
5	75° 04′ 59.850″	75° 05′ 00.431″	75° 04′ 59.850″	75° 05′ 00.431″
6	75° 02′ 59.029″	75° 02′ 58.606″	75° 02′ 58.929″	75° 02′ 58.806″
	V			
	α_i^0	$\tilde{\alpha}_i^0$		
	75° 06′ 29.325″	75° 06′ 27.806″		
	75° 03′ 39.893″	75° 03′ 39.701″		
	74° 59′ 17.603″	74° 59′ 17.334″		
	75° 00′ 21.752″	75° 00′ 26.672″		
	75° 04′ 59.863″	75° 05′ 00.532″		
	75° 02′ 58.898″	75° 02′ 58.764″		

which apparently indicates poor reliability as explained in Section 4.2. The local redundancy number κ_i is also utilised to investigate the internal reliability of the observations.

From Table **3**, the external reliability $\vartheta_i = 10.1$ can be interpreted as the influence $\left(\nabla_{E_i}\right)$ of the minimum detectable error $\left(\theta_i = 21.1"\right)$ on the coordinates of the

Table 3: Internal End External Reliability of the Angles $(\theta_i, v_i \text{ and } \sigma_i \text{ are in arc-seconds})$

Point	From	To	θ_i	ϑ_i	κ_i	v_i	σ_i	T-Test
1	2	3	1.4	3.0	45	-1.0	0.9	1.01
1	2	6	13.4	6.0	-18	-2.0	1.9	-2.35
1	6	5	3.0	2.4	355	-5.3	0.5	1.41
2	3	6	3.6	2.2	246	-0.1	1.3	-0.03
2	6	1	4.4	1.8	-165	-4.6	2.9	1.78
3	1	2	6.7	1.8	-71	-7.3	2.8	4.27
3	4	6	4.8	1.5	137	-2.4	2.4	-1.01
3	6	1	21.1	10.1	-7	-8.3	2.9	-15.42
4	5	6	4.2	1.9	179	-4.5	1.8	-1.65
4	6	3	6.7	1.8	71	-2.7	0.4	1.61
5	1	6	9.9	4.0	33	-1.0	1.3	-0.90
5	6	4	4.3	1.8	-172	-2.1	1.6	0.79
6	1	2	7.6	2.5	-55	-3.8	1.5	-2.55
6	2	3	7.4	2.3	59	-4.3	2.0	-2.78
6	3	4	3.7	2.1	-242	-3.9	4.2	1.24
6	4	5	2.6	2.5	457	-1.2	3.3	-0.28
6	5	1	7.3	2.3	60	-0.7	2.2	0.44

point i (cf. Equation 41) is 10.1 smaller than the a-posteriori standard deviation of the coordinate.

The statistical hypothesis methods stand on mathematical and stochastic models. These mathematical models state the gross error detection and correctly describe the relation between the observations and the unknown parameters while the stochastic models describe the stochastic properties of the observations. In this study T-Test is used to test the reliability of the network observations [26]. The theoretical critical value of T-Test is found to be 1.96, therefore the red-coloured values in Tables **3** and **4** represent the rejected observation because they exceed the critical value 1.96. From Table **3**, we can see that five angles in the network are rejected according to the T-Test criterion, while in Table **4**, it is clear that only pair of the observations ($\ell_{1\rightarrow5}$ and $\ell_{3\rightarrow4}$) are rejected. The minimum detectable gross errors at the rejected distances in Table **4** reach about 15cm $(\ell_{1\rightarrow5})$ and 7cm $(\ell_{3\rightarrow4})$, respectively.

Table 4: Internal end External Reliability of the Distances $(\theta_i, v_i \text{ and } \sigma_i \text{ are in meters})$

From	To	θ_i	φ_i	ϑ_i	κ_i	v_i	σ_i	T-Test
1	2	0.062	0.061	0.5	103	-0.020	0.007	-0.88
1	3	0.062	0.061	1.3	126	0.002	0.013	0.07
1	5	0.143	0.139	5.4	21	0.046	0.022	4.25
1	6	0.071	0.070	1.3	81	0.030	0.010	1.46
2	3	0.052	0.051	1.6	151	-0.019	0.016	-0.67
2	6	0.071	0.070	1.5	78	0.016	0.010	0.80
3	4	0.070	0.068	0.7	94	-0.056	0.013	-2.41
3	6	0.057	0.056	1.2	125	0.018	0.014	0.70
4	5	0.064	0.063	0.5	97	-0.003	0.012	-0.13
4	6	0.056	0.056	1.2	124	-0.027	0.006	-1.09
5	6	0.084	0.083	2.4	57	0.017	0.011	0.98

It is worth mentioning that the internal and external reliability of the observations (θ_i, ϑ_i) can only describe the reliability of the network observations. But the rejection of the observations are subject to T-Test criterion, that is why some observations in Tables **3** and **4** might seem to have good internal and external reliabilities while they are actually rejected by T-Test.

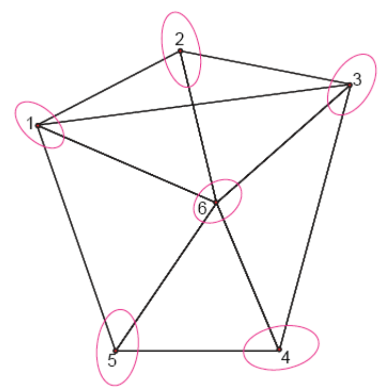


Figure 4: The final adjusted triangulation network with minimum error ellipse.

The error ellipses, also known as confidence ellipses, are considered as equivalent spots of standard deviations. There is a certain level of confidence means that a station can possibly be located within the area surrounded by its ellipse. The absolute error ellipses are computed around the benchmarks (see Figure **4**) based on their correct bearings $\bar{\alpha}_i$, the ratios between their semi-major a and semi-minor b axes are also illustrated in Table **5**. The final adjusted network including the error ellipses is plotted in Figure **4**.

6. CONCLUDING REMARKS

In this study, we used minimum-norm least-squares procedure for geodetic free network adjustment. The

Table 5: Absolute Error Ellipse Over Network Stations and their Bearings $\bar{\alpha}_i, \left(\frac{a}{b}\right)$ is the Ratio Between the Semi-major and Semi-minor Axes *a* and *b*, Respectively

Station	a / b	$\bar{\alpha}_i$
7	2.6	80
8	4.7	-23
9	2.3	-4
10	6.2	73
14	8.3	64
15	3.9	-75

free-network observations including angles and distances were compiled in the least-squares sense after the creation of the condition equations which were created by the linearisation of the observations (angles and distances).

The solution of the free network based on minimum-norm least-squares has been iterated to reduce the divergence between the initial and adjusted coordinates. The coordinates were better converged after a number of five iterations as mentioned in Section 5. The data snooping procedure has been applied after the least-squares solution, the test of reliability and precision of the observations.

Furthermore, the T-Test criterion was utilised to detect any gross errors in the observations (angles and distances) as shown in Tables **3** and **4**). The result of T-Test revealed that a number of five angles (red-coloured) are suspected to be affected by gross errors, while two lines of the network were also detected to include gross errors (cf. Tables **3** and **4**).

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