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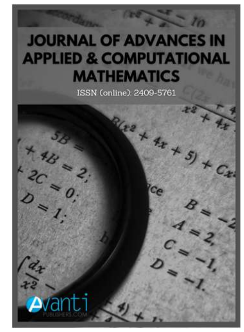

Characteristics of a Subclass of Analytic Functions Introduced by Using a Fractional Integral Operator

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ABSTRACT

We define a new class of analytic functions $D_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ on the open unit disc using the fractional integral associated with a linear differential operator and investigate characteristics of this class: extreme points, distortion bounds, radii of close-to-convexity, starlikeness and convexity.

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1. Introduction

Introducing and studying new classes of analytic functions using operators is a classical method for conducting studies regarding complex-valued functions. The operator used in the present study is obtained using fractional integral, a function that has given a great number of interesting results in the last years. A nice review on the evolution of the study related to fractional calculus can be seen in the introductory part of [1]. Many applications of fractional calculus have appeared recently. Fractional derivative operators associated with fuzzy sets theory are considered in [2, 3]. Another application of fractional calculus can be seen as new computations for the two-mode version of the fractional Zakharov-Kuznetsov model in plasma fluid by means of the Shehu decomposition method [4]. Generalized fractional integral operators are considered in the research presented in [5] and a generalized fractional integral operator is used for obtaining Hermite–Hadamard type inequality for γ-convex functions in [6]. An investigation of the sufficient conditions for the existence of solutions of two new types of coupled systems of hybrid fractional differential equations involving ϕ-Hilfer fractional derivatives is conducted in [7]. Several algebraic aspects of the fuzzy Caputo fractional derivative and fuzzy Atangana–Baleanu fractional derivative operator in the Caputo sense are investigated in [8]. Atangana-Baleanu fractional integral of Bessel functions is used for obtaining differential subordinations results in [9]. Fractional differential and integral properties of Mittag-Leffler function are studied in [10-12].

Applications of fractional integral for obtaining new operators and defining new classes have recently provided interesting outcomes as it can be seen citing papers published in the last three years [13-19].

Investigations for obtaining fuzzy differential subordinations involving different operators were investigated in recent years [20, 21]. Motivated by such results, in a recently submitted paper [22], the following operator was defined:

Definition 1.1 ([22]) Let $D^m_{n,\delta,g}: \mathcal{A} \to \mathcal{A}$ the linear differential operator defined by

$$D^0_{n,\delta,g} f(z) = (f * g)(z),$$

$$D^1_{n,\delta,g} f(z) = \left[1-(1-\delta)^n\right](f * g)(z) + (1-\delta)^n z (f * g)'(z),$$

$$D^m_{n,\delta,g} f(z) = \left[1-(1-\delta)^n\right] D^{m-1}_{n,\delta,g} f(z) + (1-\delta)^n z \left(D^{m-1}_{n,\delta,g} f(z)\right)', \quad z \in U,$$

where $m, n \in \mathbb{N}$.

Denote by $D^m_{n,\delta}: \mathcal{A} \to \mathcal{A}$,

$$D^m_{n,\delta} f(z) = D^m_{n,\delta,f} f(z).$$

If $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$, then

$$D^m_{n,\delta} f(z) = z + \sum_{j=2}^{\infty} \left[1 + (j-1)(1-\delta)^n\right]^m a_j^2 z^j, \; z \in U.$$

We remind the definition of fractional integral:

Definition 1.2 ([23]) The fractional integral of order λ ($\lambda > 0$) is defined for a function f by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.1}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.

Using Definition 1.1 and Definition 1.2, in [17] we define the fractional integral associated with the linear differential operator

$$D_z^{-\lambda} D_{n,\delta}^m f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{D_{n,\delta}^m f(t)}{(z-t)^{1-\lambda}} dt = \tag{1.2}$$

$$\frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt + \sum_{j=2}^{\infty} \frac{\left[1+(j-1)(1-\delta)^n\right]^m}{\Gamma(\lambda)} a_j^2 \int_0^z \frac{t^j}{(z-t)^{1-\lambda}} dt,$$

which can be written, after a simple calculation, by the following relation

$$D_z^{-\lambda} D_{n,\delta}^m f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \frac{\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} a_j^2 z^{j+\lambda},$$

for the function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathrm{A}$. We note that $D_z^{-\lambda} D_{n,\delta}^m f(z) \in \mathrm{A}(\lambda+1,1)$.

Following the ideas from [13] and [14] for this operator we introduce a new class of analytic functions and study several aspects regarding distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity.

The study presented in this paper is done in a well-known environment.

Denote by $U = \{z \in \mathrm{C} : |z| < 1\}$ the unit disc of the complex plane and $\mathrm{H}(U)$ the space of holomorphic functions in U.

Let $\mathrm{A}(p,t) = \{f \in \mathrm{H}(U) : f(z) = z^p + \sum_{j=p+t}^{\infty} a_j z^j,\ z \in U\}$, with $\mathrm{A}(1,1) = \mathrm{A}$ and $\mathrm{H}[a,t] = \{f \in \mathrm{H}(U) : f(z) = a + a_t z^t + a_{t+1} z^{t+1} + \ldots,\ z \in U\}$, where $p, t \in \mathrm{N}$, $a \in \mathrm{C}$.

2. Main Results

Firstly, we define the new class of analytic functions using the operator given by relation (1.2).:

Definition 2.1 The function f belongs to the class $\mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ if it satisfies the following relation:

$$\left| \frac{\lambda(1-\mu)\frac{D_z^{-\lambda} D_{n,\delta}^m f(z)}{z} + \mu(D_z^{-\lambda} D_{n,\delta}^m f(z))'}{\lambda(1-\mu)\frac{D_z^{-\lambda} D_{n,\delta}^m f(z)}{z} + \mu(D_z^{-\lambda} D_{n,\delta}^m f(z))' - \alpha} \right| < \beta, \tag{2.1}$$

where $0 < \beta \le 1$, $\lambda, \delta, \alpha, \mu > 0$, $n, m \in \mathrm{N}$, $z \in U$.

Next, we get coefficient bounds and extreme points for functions in class $\mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$.

Theorem 2.1 Consider the function $f \in \mathrm{A}$. Then $f \in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ if and only if

$$\sum_{j=2}^{\infty} \frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\Gamma(j+\lambda+1)} a_j^2 < \frac{\beta|\alpha|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}. \tag{2.2}$$

The result is sharp for the function $F(z)$ defined by

$$F(z) = z + \sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1} - \frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}}\, z^j, \quad j \ge 2. \tag{2.3}$$

Proof. Consider f satisfies (2.2). Then we obtain, for $|z|<1$,

$$\left|\lambda(1-\mu)\frac{D_z^{-\lambda}D_{n,\delta}^m f(z)}{z} + \mu(D_z^{-\lambda}D_{n,\delta}^m f(z))'\right| -$$

$$\beta\left|\lambda(1-\mu)\frac{D_z^{-\lambda}D_{n,\delta}^m f(z)}{z} + \mu(D_z^{-\lambda}D_{n,\delta}^m f(z))' - \alpha\right| =$$

$$\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda + \sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1}\right| -$$

$$\beta\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda + \sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1} - \alpha\right| \le$$

$$\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda\right| + \left|\sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1}\right| -$$

$$\beta|\alpha| + \beta\left|\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda\right| + \beta\left|\sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1}\right| <$$

$$\frac{(\beta+1)(\lambda+\mu)}{\Gamma(\lambda+2)} - \beta|\alpha| + \sum_{j=2}^{\infty}\frac{(\beta+1)(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 < 0$$

Applying the maximum modulus Theorem and (2.1), we obtain $f \in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$.

Conversely, we assume that

$$\left|\frac{\lambda(1-\mu)\frac{D_z^{-\lambda}D_{n,\delta}^m f(z)}{z}+\mu(D_z^{-\lambda}D_{n,\delta}^m f(z))'}{\lambda(1-\mu)\frac{D_z^{-\lambda}D_{n,\delta}^m f(z)}{z}+\mu(D_z^{-\lambda}D_{n,\delta}^m f(z))'-\alpha}\right|=$$

$$\left|\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1}-\alpha}\right|<\beta, z\in U.$$

Condition $Re(z)\le|z|,\ z\in U$, implies

$$Re\left\{\frac{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1}}{\frac{\lambda+\mu}{\Gamma(\lambda+2)}z^\lambda+\sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j^2 z^{j+\lambda-1}-\alpha}\right\}<\beta. \tag{2.4}$$

Considering values of z on the real axis so that $\lambda(1-\mu)\frac{D_z^{-\lambda}D_{n,\delta}^m f(z)}{z}+\mu(D_z^{-\lambda}D_{n,\delta}^m f(z))'$ is real and put $z\to 1$ through real values, we get the inequality (2.2).

Corollary 2.2 If $f\in \mathcal{A}$ be in $\mathcal{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$, then

$$a_j\le\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}},\quad j\ge 2, \tag{2.5}$$

with equality only for functions of the form $F(z)$.

Theorem 2.3 Let $f_1(z)=z$ and

$$f_j(z)=z-\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}}z^j,\quad j\ge 2, \tag{2.6}$$

for $0<\beta\le 1,\ \lambda,\delta,\alpha,\mu>0,\ n,m\in\mathbb{N}$. Then f belongs to the class $\mathcal{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ if and only if it can be written as

$$f(z)=\sum_{j=1}^{\infty}\omega_j f_j(z), \tag{2.7}$$

where $\omega_j\ge 0$ and $\sum_{j=1}^{\infty}\omega_j=1$.

Proof. Consider $f(z)$ written as in (2.7). Then

$$f(z)=z-\sum_{j=2}^{\infty}\omega_j\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}}z^j.$$

Now,

$$\sum_{j=2}^{\infty}\sqrt{\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}}\omega_j$$

$$\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}}=\sum_{j=2}^{\infty}\omega_j=1-\omega_1\leq 1.$$

Thus $f\in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$.

Conversely, let $f\in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$. Then by using (2.5), setting

$$\omega_j=\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}}a_j,\ j\geq 2$$

and $\omega_1=1-\sum_{j=2}^{\infty}\omega_j$, we obtain $f(z)=\sum_{j=1}^{\infty}\omega_j f_j(z)$. The proof of Theorem 2.3 is complete.

Distortion bounds for class $\mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ are given in the next proved result.

Theorem 2.4 If $f\in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$, then

$$r-\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}r^2\leq|f(z)| \tag{2.8}$$

$$\leq r+\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}r^2$$

holds when the sequence $\{\sigma_j(\lambda,\delta,\mu,\beta,m,n)\}_{j=2}^{\infty}$ is non-decreasing, and

$$1-\sqrt{\frac{2\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}r \le \left|f'(z)\right| \tag{2.9}$$

$$\le 1+\sqrt{\frac{2\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}r$$

holds when the sequence $\{\frac{\sigma_j(\lambda,\delta,\mu,\beta,m,n)}{j}\}_{j=2}^{\infty}$ is non- decreasing, where

$$\sigma_j(\lambda,\delta,\mu,\beta,m,n)=\sqrt{\frac{(\beta+1)(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\Gamma(j+\lambda+1)}}.$$

The bounds in (2.8) and (2.9) are sharp, for $f(z)$ given by

$$f(z)=z+\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}z^2,\ z=\pm r. \tag{2.10}$$

Proof. Using Theorem 2.1, we obtain

$$\sum_{j=2}^{\infty}a_j \le \sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}. \tag{2.11}$$

We have

$$|z|-|z|^2\sum_{j=2}^{\infty}a_j \le |f(z)| \le |z|+|z|^2\sum_{j=2}^{\infty}a_j.$$

Thus

$$r-\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}r^2 \le |f(z)| \tag{2.12}$$

$$\le r+\sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}r^2.$$

Hence (2.8) follows from (2.12).

Further,

$$\sum_{j=2}^{\infty} ja_j \le \sqrt{\frac{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(\lambda+3)}{2(\lambda+2\mu)\left[1+(1-\delta)^n\right]^m}}.$$

Hence (2.9) follows from

$$1-r\sum_{j=2}^{\infty} ja_j \le \left|f^{'}(z)\right| \le 1+r\sum_{j=2}^{\infty} ja_j.$$

In the next results, the radii of close-to-convexity, starlikeness and convexity for the class $\mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ are investigated.

Theorem 2.5 The function $f \in \mathrm{A}$ belonging to the class $\mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ is close -to-convex of order k, $0 \le k < 1$ in the disc $|z| < r$, for

$$r := \inf_{j\ge 2} \sqrt{\frac{(1-k)^2(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)j\Gamma(j+\lambda+1)}}. \tag{2.13}$$

The result is sharp, the extremal function $f(z)$ is given by (2.3).

Proof. We have to show, for $f \in \mathrm{A}$, that

$$\left|f^{'}(z)-1\right| < 1-k. \tag{2.14}$$

A simple calculation get

$$\left|f^{'}(z)-1\right| \le \sum_{j=2}^{\infty} ja_j|z|$$

and the last expression is less than $1-k$ if

$$\sum_{j=2}^{\infty}\frac{j}{1-k}a_j|z| < 1.$$

Since $f \in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ if and only if

$$\sum_{j=2}^{\infty}\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m\Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}a_j^2 < 1,$$

(2.14) holds true if

$$\frac{j}{1-k}|z| \le \sum_{j=2}^{\infty} \sqrt{\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}}.$$

Or, equivalently,

$$|z| \le \sum_{j=2}^{\infty} \sqrt{\frac{(1-k)^2(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)j\Gamma(j+\lambda+1)}},$$

which completes the proof.

Theorem 2.6 Let $f \in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$. Then

1. f is starlike of order k in the disc $|z|<r_1,\ 0\le k<1$, and

$$r_1 = \inf_{j\ge 2} \sqrt{\frac{(1-k)^2(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)(j+k-2)^2\Gamma(j+\lambda+1)}}.$$

2. f is convex of order k in the disc $|z|<r_2,\ 0\le k<1$, and

$$r_2 = \inf_{j\ge 2} \sqrt{\frac{(1-k)^2(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j-1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)j(j-1)\Gamma(j+\lambda+1)}}.$$

The results are sharp for the extremal function $f(z)$ given by (2.3).

Proof. 1. We have to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-k,\ 0\le k<1. \tag{2.15}$$

We obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \left|\frac{\sum_{j=2}^{\infty}(j-1)a_j|z|}{1+\sum_{j=2}^{\infty}a_j|z|}\right|$$

and the last expresion is less than $1-k$ if

$$\sum_{j=2}^{\infty}\frac{(j+k-2)}{1-k}a_j|z|<1.$$

Since $f \in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ if and only if

$$\sum_{j=2}^{\infty} \frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)} a_j^2 < 1.$$

(2.15) holds true if

$$\frac{j+k-2}{1-k}|z| < \sqrt{\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}}.$$

Or, equivalently,

$$|z| < \sqrt{\frac{(1-k)^2(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)(j+k-2)^2\Gamma(j+\lambda+1)}},$$

the starlikeness of the family is obtained.

2. Since f is convex if and only if zf' is starlike, we can prove (2) analogue with (1). The function f is convex if and only if

$$\left|zf''(z)\right| < 1-k. \tag{2.16}$$

We obtain

$$\left|zf''(z)\right| \le \left|\sum_{j=2}^{\infty} j(j-1)a_j z\right| < 1-k$$

$$\sum_{j=2}^{\infty} \frac{j(j-1)}{1-k} a_j |z| < 1.$$

Since $f \in \mathrm{D}_{m,n}(\lambda,\delta,\mu,\alpha,\beta)$ if and only if

$$\sum_{j=l+1}^{\infty} \frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)} a_j^2 < 1,$$

(2.16) holds true if

$$\frac{j(j-1)}{1-k}|z| < \sqrt{\frac{(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j+1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)\Gamma(j+\lambda+1)}},$$

or, equivalently,

$$|z| < \sqrt{\frac{(1-k)^2(\lambda+\mu j)\left[1+(j-1)(1-\delta)^n\right]^m \Gamma(j-1)}{\left(\frac{\beta|\alpha|}{\beta+1}-\frac{(\lambda+\mu)}{\Gamma(\lambda+2)}\right)j(j-1)\Gamma(j+\lambda+1)}},$$

which yields the convexity of the family.

3. Conclusion

A new class of analytic functions is defined in this paper using a previously introduced fractional integral operator. The class is studied regarding various characteristics such as distortion bounds and starlikeness and convexity. The results contained here could inspire further studies on the functions of this class regarding subordination and superordination results involving the fractional integral operator used in the present paper.

References

[1] Zhou SS, Rashid S, Parveen S, Akdemir AO, Hammouch Z. New computations for extended weighted functionals within the Hilfer generalized proportional fractional integral operators, AIMS Mathematics 2021; 6(5): 4507–4525. DOI: 10.3934/math.2021267.

[2] Rashid S, Ashraf R, Bayones FS. A novel treatment of fuzzy fractional Swift–Hohenberg equation for a hybrid transform within the fractional derivative operator, Fractal and Fractional 2021; 5(4): 209. https://doi.org/10.3390/fractalfract5040209.

[3] Alqudah MA, Ashraf R, Rashid S, Singh J, Hammouch Z, Abdeljawad T. Novel numerical investigations of fuzzy Cauchy reaction–diffusion models via generalized fuzzy fractional derivative operators, Fractal and Fractional, 2021; 5(4): 151. https://doi.org/10.3390/fractalfract5040151.

[4] Al-Qurashi M, Rashid S, Jarad F, Tahir M, Alsharif AM. New computations for the two-mode version of the fractional Zakharov-Kuznetsov model in plasma fluid by means of the Shehu decomposition method, AIMS Mathematics, 2022; 7(2): 2044-2060. doi:10.3934/math.2022117.

[5] Rashid S, Hammouch Z, Aydi H, Ahmad AG, Alsharif AM. Novel computations of the time-fractional Fisher's model via generalized fractional integral operators by means of the Elzaki Transform, Fractal and Fractional, 2021; 5(3): 94. https://doi.org/10.3390/fractalfract5030094.

[6] Rashid S, Khalid A, Bazighifan O, Oros GI. New modifications of integral inequalities via γ -Convexity pertaining to fractional calculus and their applications, Mathematics 2021; 9(15): 1753; https://doi.org/10.3390/math9151753.

[7] Almalahi MA, Bazighifan O, Panchal SK, Askar SS, Oros GI. Analytical study of two nonlinear coupled hybrid systems involving generalized Hilfer fractional operators, Fractal and Fractional, 2021; 5(4): 178; https://doi.org/10.3390/fractalfract5040178.

[8] Rashid S, Ashraf R, Akdemir AO, Alqudah MA, Abdeljawad T, Mohamed MS. Analytic fuzzy formulation of a time-fractional Fornberg–Whitham model with Power and Mittag–Leffler kernels, Fractal and Fractional, 2021; 5(3): 113. https://doi.org/10.3390/fractalfract5030113.

[9] Lupas AA, Catas A. An application of the principle of differential subordination to analytic functions involving Atangana-Baleanu fractional integral of Bessel functions, Symmetry, 2021; 13: 971. https://doi.org/10.3390/sym13060971.

[10] Srivastava HM, Bansal M, Harjule P. A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function, Math. Meth. Appl. Sci., 2018; 1-14.

[11] Ghanim F, Al-Janaby HF. An analytical study on Mittag-Leffler-confluent hypergeometric functions with fractional integral operator. Math. Methods Appl. Sci., 2021; 44(5): 3605-3614.

[12] Ghanim F, Al-Janaby HF, Bazighifan O. Some new extensions on fractional differential and integral properties for Mittag-Leffler confluent hypergeometric function, Fractal and Fractional, 2021; 5: 143.

[13] Lupas AA, Cioban M. On a subclass of analytic functions defined by a fractional integral operator, International Conference on Mathematics, Informatics and Information Technologies, 19-21 April, 2018; Bălți, Republica Moldova, 8-14.

[14] Lupas AA. A subclass of analytic functions defined by a fractional integral operator, J. Comput. Anal. Appl., 2019; 27(3): 502-505.

[15] Cho NE, Aouf MK, Srivastava R. The principle of differential subordination and its application to analytic and p-valent functions defined by a generalized fractional differintegral operator, Symmetry 2019; 11: 1083.

[16] Lupas AA. Inequalities for Analytic Functions Defined by a Fractional Integral Operator. In Frontiers in Functional Equations and Analytic Inequalities; Anastassiou, G., Rassias, J., Eds.; Springer: Berlin/Heidelberg, Germany, 2020; 731-745.

[17] Lupas AA. About a subclass of analytic functions defined by a fractional integral operator, Montes Taurus J. Pure Appl. Math. 2021; 3(3): 200-210.

[18] Lupas AA. New Applications of the Fractional Integral on Analytic Functions, Symmetry 2021; 13(3): 423; https://doi.org/10.3390/sym13030423

[19] Lupas AA, Oros GI. Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function, Symmetry 2021; 13(2): 327; https://doi.org/10.3390/sym13020327

[20] Lupas AA, Oros GI. On special differential subordinations using fractional integral of Să lă gean and Ruscheweyh operators, Symmetry 2021; 13(9): 1553. https://doi.org/10.3390/sym13091553

[21] Oros GI. Fuzzy differential subordinations obtained using a hypergeometric integral operator, Mathematics, 2021; 9(20): 2539. https://doi.org/10.3390/math9202539

[22] El-Deeb SM, Lupas AA. Fuzzy Differential Subordinations Connected with Convolution, Studia Universitatis Babe S -Bolyai Mathematica, accepted 2020.

[23] Cho NE, Aouf AMK. Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients, Tr. J. of Mathematics, 1996; 20: 553-562.