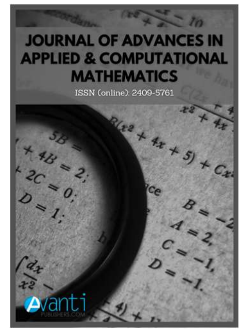
Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761



Analysis of the Oscillations of Stratified Liquid with Elastic Ice

D.O. Tsvetkov*

V.I. Vernadsky Crimean Federal University, Russia

ARTICLE INFO

Article Type: Research Article

Keywords:
Strong solution
Cauchy problem
Spectral problem
Initial-boundary value problem
Differential equation in Hilbert space

Timeline:
Received: August 28, 2021
Accepted: October 27, 2021
Published: December 31, 2021

Citation: Tsvetkov DO. Analysis of the Oscillations of Stratified Liquid with Elastic Ice. J Adv App Comput Math. 2021; 8: 87-97.

DOI: https://doi.org/10.15377/2409-5761.2021.08.6

ABSTRACT

We study the problem of small motions of an ideal stratified liquid with a free surface totally covered by an elastic ice. The elastic ice is modeled by an elastic plate. We reduce the original initial boundary value problem to an equivalent Cauchy problem for a second-order differential equation in a Hilbert space. We obtain conditions under which there exists a strong (with respect to time) solution of the initial boundary value problem describing the evolution of the hydrodynamic system under consideration. We also study the spectrum of normal oscillations, the basic properties of the eigenfunctions.

MSC: 35D35, 35Q35, 35P05, 76B70

*Corresponding Author
Email: tsvetdo@gmail.com
Tel: +79787293074

1. Introduction

In connection with the needs of applied sciences, an interest in the study of dynamic characteristics of liquids with different specific properties has increased. Such liquids are, in particular, stratified. The interest is due to not only practical needs but also to the theoretical content of the arising problems. In many cases, mathematical models for such problems are essentially nonlinear and could be investigated by numeric methods, only. But some interesting and useful problems can be investigated in terms of linear models leading to non-traditional initial-boundary value problems. This defines an independent mathematical interest in such problems.

By stratified liquid we mean a liquid whose physical characteristics (density, heat capacity, dynamical viscosity, etc.), in the stationary state, change either continuously or abruptly in only one selected direction. In other words, in the stationary state, the physical characteristics of the liquid are functions of a unique space variable. Stratification of liquid occurs due to different physical reasons; the most common of them is gravitation. In the liquid, gravitation produces such distribution of its particles, dissolved salts, and suspensions that inhomogeneity of liquid density along the direction of the gravitational field arises. As the experiment shows, stratification of density has the most significant impact on the dynamical properties of liquid and the distribution of its inner waves. Therefore, further by stratified liquid we will mean a liquid with stratification of density, induced by gravitation.

To date, various analytical and numerical approaches have been developed to study the behaviour of various structures interacting with a liquid [1-7]. At the same time, a number of methods based on functional analysis and the theory of operators in abstract Hilbert spaces are known in mathematical physics. As an example, we can cite the theory of quadratic operator beams, which is widely used in the study of various problems of mechanics and hydrodynamics related to the problem of small motions and normal oscillations (see, for example, [8-18]).

The paper is a continuation of the papers [11-13], in which initial boundary value problems of the dynamics of a liquid covered with ice were studied. Namely, in work [11], the problem of small movements of an ideal homogeneous liquid was studied, the free surface of which consists of three regions: the surface of a liquid without ice, the region of elastic ice and the region of crumbling ice. As well as the most general formulation of the problem, when on the free surface there are regions of the elastic ace of different density with different stiffness coefficient and regions of crushed ice different densities. The elastic ice is modelled by an elastic plate. The crumbled ice is understood as weighty particles of some matter floating on the free surface. The conditions under which there is a strong (in time) solution of the initial boundary value problem are obtained, and the spectrum of normal oscillations is studied. A similar problem of small movements of a stratified fluid (in the general case) is studied in [12]. In [13], a special case was considered when the free surface is completely covered with elastic ice. The presence of stratification complicates the corresponding problem from [11], namely, the structure of the operator coefficients has a more difficult structure to study. The conclusion of the conditions for the existence of the solution was carried out formally.

Thus, the purpose of this work is to present the results obtained, related to the derivation of the basic conditions for the existence of a solution from work [13], as well as study of the spectrum and properties of eigenfunctions (the problem of normal oscillations).

2. Mathematical Statement of the Problem

We consider an ideal stratified liquid whose density ρ_0 varies along the vertical axis Ox_3 in the quiescent state, $\rho_0 = \rho_0(x_3)$. The liquid fills partially a stationary vessel and in the quiescent state, it fills a domain Ω bounded by a hard wall S and a free surface Γ totally covered by elastic ice. We denote the surface density of the elastic ice by ρ_1. We assume that the origin O of the Cartesian coordinate system $Ox_1x_2x_3$ is taken on a free equilibrium surface Γ which is flat and is located perpendicular to the acceleration of the gravity $\vec{g} = -g\vec{e}_3$, where $\vec{e}_3$ is the unit vector of the axis Ox_3. We also assume that the hard wall $S \subset \partial\Omega$ is a Lipschitz surface and $\partial S = \partial\Gamma$ is a Lipschitz curve. We consider the main case of a stable stratification of the liquid with respect to the density:

$$0 < N_{min}^2 \le N^2(x_3) \le N_{max}^2 = N_0^2 < \infty, \quad N^2(x_3) = -\frac{g\rho_{0'}(x_3)}{\rho_0(x_3)}, \quad \rho_0(0) > 0, \tag{1}$$

The function $N(x_3)$ is called Brunt-Vaisala frequency or buoyancy frequency. Physically, $N(x3)$ is equal to the frequency of oscillations made by a particle of a liquid at $x_3 = \text{const}$ in the stratified liquid once this particle is moved from this level.

Consider small motions close to the state at rest. Let $\vec{u} = \vec{u}(t,x)$, $x = (x_1, x_2, x_3) \in \Omega$, we denote the velocity field of the liquid, by $p = p(t,x)$, the deviation of the pressure field from the equilibrium pressure $P_0 = P_0(x_3)$, by $\rho = \rho(t,x)$, the deviations of the density field from the original field $\rho_0(x_3)$, and by $\zeta = \zeta(t,\hat{x})$, $\hat{x} \in \Gamma$, the deviation of the freely moving surface $\Gamma(t)$ of the liquid from Γ along the normal $\vec{n}$. In addition, we believe that a small field of external forces $\vec{f}(t,x)$ acts on the hydrodynamic system under study in addition to the gravitational field.

The linear statement of the initial boundary value problem of oscillations of the hydrodynamic system under study has the form (see [13])

$$\begin{aligned}
&\frac{\partial \vec{u}}{\partial t} = \rho_0^{-1}(x_3)(-\nabla p - g\rho\vec{e}_3) + \vec{f}(t,x) \quad (\text{in}\,\Omega),\\
&\text{div}\vec{u} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla\rho_0 \cdot \vec{u} = 0 \quad (\text{in}\,\Omega),\\
&\vec{u}\cdot\vec{n} =: u_n = 0 \quad (\text{on}\,S), \quad u_n = \frac{\partial \zeta}{\partial t} \quad (\text{on}\,\Gamma),\\
&p = \rho_1 \frac{\partial^2 \zeta}{\partial t^2} + K\zeta \quad (\text{on}\,\Gamma), \quad \int_\Gamma \zeta\, d\Gamma = 0,\\
&\vec{u}(0,x) = \vec{u}^0(x), \quad \rho(0,x) = \rho^0(x) \quad (x\in\Omega), \quad \zeta(0,\hat{x}) = \zeta_0(\hat{x}) \quad (\hat{x}\in\Gamma).
\end{aligned} \tag{2}$$

The last two conditions are the initial conditions added to the problem for the completeness of its statement, $\int_\Gamma \zeta\, d\Gamma = 0$ is the condition of the preservation of volume.

Here K is the linear differential operator defined by the differential expression:

$$K\zeta := d\Delta_2^2\zeta + \rho_0(0)g\zeta$$

on the domain $\mathsf{D}(K) = \left\{\zeta \in C^4\left(\overline{\Gamma}\right): \zeta = \partial\zeta/\partial\nu = 0(\partial\Gamma)\right\}$, where $d > 0$ is the coefficient of rigidity of ice, ν is the unit vector of the outward normal to $\partial\Gamma$ and $\Delta_2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. Moreover, the operator $K : \mathsf{D}(K) \subset L_2(\Gamma) \to L_2(\Gamma)$ (to be more precise, it's Friedrichs extension) is an unbounded self-adjoint positive definite operator (see, e.g., [13]).

In the initial boundary value problem (2), we can eliminate one unknown function, namely, the density field $\rho(t,x)$, by replacing the velocity field $\vec{u}(t,x)$ with the field $\vec{v}(t,x)$ of small displacements of liquid particles, which is related to $\vec{u}(t,x)$ as

$$\frac{\partial \vec{v}}{\partial t} = \vec{u}, \quad \operatorname{div}\vec{v} = 0 \quad (\text{in}\Omega). \tag{3}$$

Then we obtain the relations

$$\begin{aligned} &\rho(t,x) = -\nabla\rho_0 \cdot \vec{v}(t,x) + f_0(x) = -\rho_0'(x_3)v_3(t,x) + f_0(x), \\ &f_0(x) := \rho(0,x) + \rho_0'(x_3)v_3(0,x), \quad v_3 := \vec{v}\cdot\vec{e}_3, \end{aligned} \tag{4}$$

In view of the above considerations, we rewrite original problem (2) as

$$\begin{aligned} &\frac{\partial^2 \vec{v}}{\partial t^2} = -\rho_0^{-1}(x_3)\nabla p - N^2(x_3)v_3\vec{e}_3 + \psi_0(x), \quad \operatorname{div}\vec{v} = 0 \quad (\text{in}\Omega), \\ &\vec{v}\cdot\vec{n} =: v_n = 0 \quad (\text{on}S), \quad p = \rho_1 \frac{\partial^2 v_3}{\partial t^2} + Kv_3 \quad (\text{on}\Gamma), \\ &\frac{\partial \vec{v}}{\partial t}(0,x) = \vec{u}(0,x) = \vec{u}^0(x), \quad \vec{v}(0,x) = \vec{v}^0(x), \\ &v_3(0,\hat{x}) = \zeta(0,\hat{x}) = \zeta^0(\hat{x}) \quad (\hat{x}\in\Gamma). \end{aligned} \tag{5}$$

where $\psi_0(x) := \vec{f}(t,x) - gf_0(x)\vec{e}_3/\rho_0(x_3)$.

The initial boundary value problem (5) contains only two unknown functions: the vector field $\vec{v}(t,x)$ and the scalar field $p(t,x)$ of pressures. Knowing a solution $\vec{v}(t,x)$ of the problem (5), we can determine solutions $\vec{u}(t,x)$ and $\rho(t,x)$ of the problem (2) by formulas (3) and (4).

Note that in [13], problem (5) was reduced to the Cauchy problem for the differential second-order equation in Hilbert space $\mathsf{H} = \mathsf{H}_1 \oplus \mathsf{H}_2$ of the following form

$$\frac{d^2}{dt^2}\mathsf{C}y + \mathsf{B}_K y = g(t), \qquad y(0) = y^0, \qquad y'(0) = y^1, \tag{6}$$

where

$$\mathsf{C} = \begin{pmatrix} I_1 & 0 \\ 0 & C + \rho_1 I_2 \end{pmatrix}, \quad \mathsf{B}_K = \mathsf{B}_C + \mathsf{K}_0 := \begin{pmatrix} B_{11} & B_{12}C^{1/2} \\ C^{1/2}B_{21} & C^{1/2}B_{22}C^{1/2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & K_0 \end{pmatrix},$$

$y = y(t) = (y_1; y_2)^t$ is an unknown function, $g = g(t) = (g_1; g_2)^t$ is a given function, I_i $(i = 1,2)$ are the identity operators acting in the spaces H_i, respectively.

The conditions are holding for the operator coefficients

$$0 \ll K_0 = K_0^*, \qquad \overline{D(K_0)} = H_2, \qquad 0 \le B_C = B_C^* \in L(H), \qquad 0 < C = C^* \in S_\infty(H_2),$$

L(H) and $S_\infty(H_2)$ are the space of bounded operators and the space of compact operators, respectively.

3. An Existence Theorem for the Strong Solution

Let's rewrite equation (6) in the following form:

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ (C+\rho_1 I_2)y_2 \end{pmatrix} + \begin{pmatrix} I_1+B_{11} & B_{12}C^{1/2} \\ C^{1/2}B_{21} & C^{1/2}B_{22}C^{1/2}+K_0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{7}$$

We make the change $K_0^{1/2}y_2 = w_2$ in (7), then we apply the operator $\mathrm{diag}(I_1; K_0^{-1/2})$ to both sides of the equation, as a result , we come to the equation

$$\frac{d^2}{dt^2}\begin{pmatrix} y_1 \\ K_0^{-1/2}(C+\rho_1 I_2)K_0^{-1/2}w_2 \end{pmatrix} + \begin{pmatrix} I_1+B_{11} & B_{12}C^{1/2}K_0^{-1/2} \\ K_0^{-1/2}C^{1/2}B_{21} & K_0^{-1/2}C^{1/2}B_{22}C^{1/2}K_0^{-1/2}+I_2 \end{pmatrix}\begin{pmatrix} y_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ K_0^{-1/2}g_2 \end{pmatrix} + \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ w_2 \end{pmatrix}.$$

Now let $K_0^{-1/2}(C+\rho_1 I_2)K_0^{-1/2}w_2 = K_0^{-1/2}A_C K_0^{-1/2}w_2 = z_2$, which is equivalent to

$$A_{K_0}w_2 := K_0^{-1/2}A_C K_0^{-1/2}w_2 = K_0^{-1/2}A_C K_0^{-1/2}(K_0^{1/2}y_2) = K_0^{-1/2}A_C y_2 = z_2.$$

Thus we get the following Cauchy problem:

$$\frac{d^2 z}{dt^2} + \mathsf{A}z = \hat{g} + Rz, \quad z(0) = (y_1(0); z_2(0))^t, \quad z'(0) = (y_1'(0); z_2'(0))^t, \tag{8}$$

$$\mathsf{A} := I_B F, \quad R = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{g} = (g_1; K_0^{-1/2}g_2)^t, \quad z = (y_1; z_2)^t,$$

$$I_B = \begin{pmatrix} I_1 + B_{11} & B_{12}C^{1/2}K_0^{-1/2} \\ K_0^{-1/2}C^{1/2}B_{21} & K_0^{-1/2}C^{1/2}B_{22}C^{1/2}K_0^{-1/2} + I_2 \end{pmatrix}, \qquad F = \begin{pmatrix} I_1 & 0 \\ 0 & A_{K_0}^{-1} \end{pmatrix}.$$

where I_B is a linear bounded self-adjoint positive definite operator, $\mathsf{D}(I_B F) = \mathsf{D}(F)$.

Let us introduce the following equivalent norm in the space H: $[v_1; v_2] := (I_B^{-1}v_1; v_2)$, then

$$[I_B F v_1; v_2] = (F v_1; v_2) = (v_1; F v_2) = (v_1; I_B^{-1} I_B F v_2) = [v_1; I_B F v_2],$$

consequently, $\mathsf{A} = I_B F$ is a self-adjoint operator, moreover, it is unbounded and positively defined.

Definition 1 By a strong (by the variable t) solution to problem (8) on $[0;T]$ we call a function $z(t)$, taking values in H, such that:

1. $z(t) \in \mathsf{D}(\mathsf{A})$ for every $t \in [0;T]$ and $\mathsf{A}z(t) \in C([0;T];\mathsf{H})$,
2. $z(t) \in C^2([0;T];\mathsf{H})$,
3. the equality (8) takes place where all summands are from $C([0;T];\mathsf{H})$, and the initial conditions hold.

Theorem 1 If the conditions

$$z(0) \in \mathsf{D}(F), \quad z'(0) \in \mathsf{D}(F^{1/2}), \quad \hat{g} \in C^1([0;T];\mathsf{H}), \tag{9}$$

take place, then problem (8) has a unique strong solution on $[0;T]$.

Proof. Since the operator $\mathsf{A} = I_B F$ is self-adjoint and positive definite in a space with an equivalent norm, therefore it is a generator of a family of cosine-functions acting in this space (see [19, pp. 175-177]). Further, since the operator R from (8) is bounded, the operator $\mathsf{A} - R = I_B F - R$, according to Theorem 8.5 from [19, p.177], is also a generator of a family of cosine-functions. It follows that if the conditions (9) are met, the problem (8) has a unique strong solution on $[0;T]$ (Definition 1). □

Let the conditions (9) take place, then the Cauchy problem (8) has a unique strong solution on $[0;T]$ (Theorem 1). Then, taking into account the changes $A_{K_0} w_2 = z_2$ and $K_0^{1/2} y_2 = w_2$, we have

$$\left(z^0 = (y_1^0; z_2^0)^t \in \mathsf{D}(F)\right) \Leftrightarrow \left(y_1^0 \in H_1,\quad z_2^0 \in \mathsf{D}(A_{K_0}^{-1})\right) \Leftrightarrow$$

$$\Leftrightarrow \left(y_1^0 \in H_1,\quad A_{K_0}^{-1} z_2^0 = A_{K_0}^{-1} A_{K_0} K_0^{1/2} y_2^0 \in H_2\right) \Leftrightarrow$$

$$\Leftrightarrow \left(y_1^0 \in H_1,\quad y_2^0 \in \mathsf{D}(K_0^{\ 1/2})\right).$$

Further,

$$\left(z^1 = (y_1^1; z_2^1)^t \in \mathsf{D}(F^{1/2})\right) \Leftrightarrow \left(u_1^1 \in H_1,\quad z_2^1 \in \mathsf{D}(A_{K_0}^{-1/2})\right) \Leftrightarrow$$

$$\Leftrightarrow \left(y_1^1 \in H_1,\quad A_{K_0}^{-1/2} z_2^1 = A_{K_0}^{-1/2} A_{K_0} K_0^{\ 1/2} y_2^1 = A_{K_0}^{1/2} K_0^{\ 1/2} y_2^1 \in H_2\right) \Leftrightarrow$$

$$\Leftrightarrow \left(y_1^1 \in H_1,\quad y_2^1 \in \mathsf{D}(K_0^{\ 1/2})\right).$$

$$\hat{g}(t) \in C^1([0;T];\mathsf{H}) \Leftrightarrow \left(g_1 \in C^1([0;T];H_1), K_0^{\ -1/2} g_2 \in C^1([0;T];H_2)\right) \Leftrightarrow$$

$$\Leftrightarrow \left(g_1 \in C^1([0;T];H_1),\quad g_2 \in C^1([0;T];H_2)\right).$$

Here the operator $K_0^{\ -1/2}$ is a bounded operator. Thus, the following theorem is proved

Theorem 2 If the conditions

$$y^0 \in H_1 \oplus \mathsf{D}(K_0^{1/2}),\quad u^1 \in H_1 \oplus \mathsf{D}(K_0^{1/2}),\quad g(t) \in C^1([0;T];\mathsf{H}), \tag{10}$$

take place, then problem (6) has a unique strong solution on $[0;T]$.

The rest of the proof is based on the reverse transition from the Cauchy problem (6) to the initial boundary value problem (5) and then to a problem (2). Thus, the Cauchy problem (2) has a unique solution, strong with respect to t, under the initial conditions of Theorem 1 of the work [13].

4. Spectral Problem

In the absence of external forces (except for the gravitational field), i.e., for $g(t,x) = 0$, consider the eigen-oscillations, which are solutions to problem (6) depending on time by the law $\exp(i\omega t): y(t,x) = e^{i\omega t} y(x)$. For the amplitude elements $y = y(x)$, we obtain the spectral problem

$$\lambda \mathsf{C} y = \mathsf{B}_K y, \quad \lambda := \omega^2. \tag{11}$$

Since $\mathsf{B}_K = \mathsf{B}_K \geq 0$ and $0 \leq \mathsf{C}^{-1} = (\mathsf{C}^{-1})^* \in \mathsf{L}(\mathsf{H})$ then the spectrum of the problem (11) is real and positive. Note that in the case when an ideal stratified liquid completely fills an arbitrary vessel, the corresponding spectral problem can be reduced to the problem $B_{11} y_1 = \lambda y_1$, $y_1 \in H_1$. In this case, there is a point spectrum, dense on the segment $[0; N_0^2]$, and the modes of eigen-oscillations give internal waves due to the presence of a stratified liquid.

4.1. About the Existence of Internal Waves

Consider the case $\lambda \in [0; N_0^2]$ (see (1)) and establish the presence of internal waves in a stratified fluid.

Let's write equation (11) in components and rewrite it in the following form

$$\begin{cases} \lambda I_1 y_1 = B_{11} y_1 + B_{12} C^{1/2} y_2, \\ -C^{1/2} B_{21} y_1 = \left(-\lambda \left(C + \rho_1 I_2\right) + C^{1/2} B_{22} C^{1/2} + K_0\right) y_2 =: M(\lambda) y_2. \end{cases} \tag{12}$$

Let's make an assumption

$$N_0^2 \left(C + \rho_1 I_2\right) < C^{1/2} B_{22} C^{1/2} + K_0, \tag{13}$$

then the operator-function $M(\lambda)$ for any $\lambda \in [0; N_0^2]$ is positively defined, therefore, for these λ there is an inverse operator $M^{-1}(\lambda) \in \mathsf{L}(\mathsf{H})$.

We express the value of y_2 from the second equation of the system (12) and substitute it into the first one, we get

$$T(\lambda) y_1 := \left(\lambda I_1 - B_{11} + B_{12} C^{1/2} M^{-1}(\lambda) C^{1/2} B_{21}\right) y_1 = 0, \qquad \lambda \in [0; N_0^2]. \tag{14}$$

Theorem 3 The limiting spectrum of the pencil $T(\lambda)$ coincides with the segment $[0; N_0^2]$.

Proof. Recall that the limit spectrum of an operator is the set of the points of the continuous spectrum, the limit points of the point spectrum, and the eigenvalues of infinite multiplicity.

Let's fix an arbitrary $\lambda_1 \in [0; N_0^2]$ and consider the problem

$$\left(\lambda I_1 - B_{11} + B_{12} C^{1/2} M^{-1}(\lambda_1) C^{1/2} B_{21}\right) y_1 = 0, \qquad y_1 \in H_1.$$

This eigenvalue problem for a self-adjoint operator B_{11} is perturbed by a compact operator $B_{12} C^{1/2} M^{-1}(\lambda_1) C^{1/2} B_{21}$. The spectrum of operator B_{11} is the limit and fills the entire segment $[0; N_0^2]$. According to Weyl's theorem, for every $\lambda_2 \in [0; N_0^2]$ there is an orthonormal sequence Weyl $\{y_i\}_{i=1}^{\infty}$, depending on λ_1 and λ_2, for which

$$\left\| \left(\lambda_2 I_1 - B_{11} + B_{12} C^{1/2} M^{-1}(\lambda_1) C^{1/2} B_{21}\right) y_i \right\| \to 0 \qquad (i \to \infty).$$

Choosing $\lambda_2 = \lambda_1$ and the corresponding Weyl sequence, we conclude that for it

$$\left\|\left(\lambda_1 I_1 - B_{11} + B_{12} C^{1/2} M^{-1}(\lambda_1) C^{1/2} B_{21}\right) y_i\right\| \to 0 \qquad (i \to \infty).$$

This means that the arbitrarily selected point $\lambda_1 \in [0; N_0^2]$ belongs to the limit spectrum of the problem (14). Since the points lying outside the segment $[0; N_0^2]$, can only be finite-multiplicity eigenvalues, the specified segment coincides with the limiting spectrum of the pencil $T(\lambda)$. □

An important consequence of the obtained theorem is the following statement: in a stably stratified ideal liquid partially filling a vessel of arbitrary shape with elastic ice on a free surface, there are internal waves due to the presence of buoyancy forces; the square of the frequencies of internal waves form the set $[0; N_0^2]$.

4.2. On the Properties of Surface Wave Modes

Consider the case $\lambda > N_0^2$ when surface waves are expected. Let's replace $y_2 = K_0^{-1/2} z_2$ and applying the operator $K_0^{-1/2}$ to the second equation (12), we obtain

$$\begin{cases} \left(I_1 - \lambda^{-1} B_{11}\right) y_1 - \lambda^{-1} B_{12} C^{1/2} K_0^{-1/2} z_2 = 0, \\ I_2 z_2 - \lambda K_0^{-1/2} \left(C + \rho_1 I_2\right) K_0^{-1/2} z_2 + K_0^{-1/2} C^{1/2} B_{21} y_1 + K_0^{-1/2} C^{1/2} B_{22} C^{1/2} K_0^{-1/2} z_2 = 0. \end{cases}$$

Since $\lambda > N_0^2$ and $\|B_{11}\| \leq N_0^2$ (see [13]), then the operator $I - \lambda^{-1} B_{11}$ is invertible. Let's rewrite the last system

$$\begin{cases} I_1 y_1 - \lambda^{-1} (I - \lambda^{-1} B_{11})^{-1} B_{12} C^{1/2} K_0^{-1/2} z_2 = 0, \\ I_2 z_2 - \lambda K_0^{-1/2} \left(C + \rho_1 I_2\right) K_0^{-1/2} z_2 + K_0^{-1/2} C^{1/2} B_{21} y_1 + K_0^{-1/2} C^{1/2} B_{22} C^{1/2} K_0^{-1/2} z_2 = 0. \end{cases}$$

Excluding y_1, we arrive at the spectral problem for the operator pencil $L(\lambda)$:

$$L(\lambda) z_2 := \left(I - \lambda B_C + B_0 + \lambda^{-1} F(\lambda)\right) z_2 = 0, \qquad \lambda > N_0^2, \tag{15}$$

$$B_C := K_0^{-1/2} (C + \rho_1 I_2) K_0^{-1/2}, \qquad B_0 := K_0^{-1/2} C^{1/2} B_{22} C^{1/2} K_0^{-1/2},$$

$$F(\lambda) := K_0^{-1/2} C^{1/2} B_{21} R(\lambda) B_{12} C^{1/2} K_0^{-1/2}, \qquad R(\lambda) := (I - \lambda^{-1} B_{11})^{-1}.$$

Further research is based on the idea of factorization of the pencil $L(\lambda)$, i.e. on its decomposition into operator multipliers of a certain type. To do this, we will need the following result (see [17, p.69-75]).

Theorem 4 Let the conditions

1. $\exists t \in (0; r): \|A\| t^{-1} + \sum_{k=1}^{\infty} \|B_k\| t^{k-1} < 1$

2. $B(\mu) := \sum_{k=1}^{\infty} \mu^k B_k$, $|\mu| < r, \quad 0 < r < \infty$; $A = A^*$, $B_k = B_k^*$, $k = 1, 2, \ldots$

be satisfied for the operator a self-adjoint operator pencil

$$M(\mu) := \mu I - A - B(\mu). \tag{16}$$

Then

1. $M(\mu)$ admits the factorization $M(\mu) = M_+(\mu)(\mu I - Z)$, where $M_+(\mu)$ is holomorphic and holographically invertible for $|\mu| \le t$, $t \in (0,r)$, and $\sigma(Z) \subset (-t;t)$, operator Z is similar to a self-adjoint operator.

2. If the conditions $A \in \mathfrak{S}_\infty(\mathrm{H})$, $\ker A = \{0\}$, $B_1 \in \mathfrak{S}_\infty(\mathrm{H})$ are additionally hold, then the problem $M(\mu)z = 0$ has a discrete spectrum on the interval $(-t,t)$

$$\sigma(Z) = \{0\} \cup \{\mu_j\}_{j=1}^\infty, \quad \mu_j \to 0 \quad (j \to \infty),$$

where $\mu_j = \mu_j(Z)$ is the isolated finite multiplicity eigenvalues of the operator Z. The set of eigenelements (there are no associated) $\{\varphi_j\}_{j=1}^\infty \subset \mathrm{H}$ corresponding to these eigenvalues forms the Riesz basis in H: $\varphi_j = F^{1/2} z_j$, $j = 1,2,\ldots$, where $\{z_j\}_{j=1}^\infty$ is an orthonormal basis consisting of elements of a self-adjoint compact operator $F^{-1/2}(ZF)F^{-1/2}$.

After substitution $\lambda = \mu^{-1}$ in (15) we obtain

$$G(\mu)z_2 := \mu L(\mu^{-1})z_2 = \left(\mu I - B_C - B(\mu)\right)z_2 = 0, \tag{17}$$

$$B(\mu) := -\mu B_0 - \mu^2 F(\mu^{-1}), \quad \mu < N_0^{-2}.$$

Problem (17) is a problem for a pencil of the form (16), since $F(\mu^{-1})$ is a holomorphic function with respect to μ:

$$F(\mu^{-1}) = \sum_{k=0}^{\infty} \mu^k F_k, \quad F_k = K_0^{-1/2} C^{1/2} B_{21} B_{11}^k B_{12} C^{1/2} K_0^{-1/2}.$$

At the same time

$$\begin{aligned} B(\mu) \quad &:= -\mu B_0 - \mu^2 F(\mu^{-1}) = -\mu B_0 - \mu^2 \sum_{k=0}^{\infty} \mu^k F_k = -\mu K_0^{-1/2} C^{1/2} B_{22} C^{1/2} K_0^{-1/2} - \\ &-\mu^2 K_0^{-1/2} C^{1/2} B_{21} B_{12} C^{1/2} K_0^{-1/2} - \mu^3 K_0^{-1/2} C^{1/2} B_{21} B_{11}^2 B_{12} C^{1/2} K_0^{-1/2} - \cdots = \\ &=: -\sum_{k=1}^{\infty} \mu^k B_k, \qquad \text{where} \quad \|B_k\| \le (N_0^2)^k \cdot \|K_0^{-1}\| \cdot \|C\|. \end{aligned}$$

Lemma 1 If $|\mu| = t < N_0^{-2}$ for $G(\mu)$ then

$$\|B_C\| \cdot t^{-1} + \sum_{k=1}^{\infty} \|B_k\| \cdot t^{k-1} < \frac{\rho_m \|K_0^{-1}\|}{t} + \frac{\|K_0^{-1}\| \cdot \|C\|}{t \cdot (1 - tN_0^2)}.$$

Proof.

$$\|B_C\| t^{-1} + \sum_{k=1}^{\infty} \|B_k\| t^{k-1} = \frac{\|K_0^{-1/2}(C + \rho_1 I_2)K_0^{-1/2}\|}{t} + \|B_1\| + \|B_2\| t + \|B_3\| t^2 + \cdots \le$$

$$\leq \frac{\|K_0^{-1}\| \cdot \|C\| + \rho_1\|K_0^{-1}\|}{t} + N_0^2 \cdot \|K_0^{-1}\| \cdot \|C\| + (N_0^2)^2 \cdot \|K_0^{-1}\| \cdot \|C\|t + \cdots =$$

$$= \frac{\rho_1\|K_0^{-1}\|}{t} + \frac{\|K_0^{-1}\| \cdot \|C\|}{t} \cdot (1 + N_0^2 t + (N_0^2)^2 t^2 + \cdots) =$$

$$= \frac{\rho_1\|K_0^{-1}\|}{t} + \frac{\|K_0^{-1}\| \cdot \|C\|}{t \cdot (1 - tN_0^2)}. \qquad \square$$

A corollary of Lemma 1 and Theorem 4 is the lemma.

Lemma 2 Let the following condition hold true

$$D := (\rho_1 \cdot \|K_0^{-1}\| \cdot N_0^2 - 1)^2 - 4 \cdot N_0^2 \cdot \|K_0^{-1}\| \cdot \|C\| > 0. \tag{18}$$

Then $G(\mu)$ from (17) admits the factorization

$$G(\mu) = G_+(\mu)(\mu I - Z), \quad |\mu| < t \in (t_-; t_+),$$
$$t_\pm := \frac{(\rho_1 \cdot \|K_0^{-1}\| \cdot N_0^2 + 1) \pm \sqrt{D}}{2N_0^2}, \quad t_+ < N_0^{-2}.$$

where $G_+(\mu)$ is holomorphic and holographically invertible for $|\mu| \leq t \in (t_-; t_+)$, and $\sigma(Z) \subset (-t; t)$.

The obtained facts allow us to prove the following theorem.

Theorem 5 If condition (18) take place, then $L(\lambda)$ from (15) admits the factorization

$$L(\lambda) = L_+(\lambda)(I - \lambda Z);$$

where $L_+(\lambda)$ is holomorphic and holographically invertible for

$$\lambda \geq (t_-)^{-1} > N_0^2, \quad t_- = \frac{(\rho_1 \cdot \|K_0^{-1}\| \cdot N_0^2 + 1) - \sqrt{D}}{2N_0^2},$$

and problem (15) has a discrete spectrum $\{\lambda_k\}_{k=1}^{\infty} \subset R_+$, $\lambda_k = [\lambda_k(Z)]^{-1}$ consisting of isolated finite multiplicity eigenvalues with a limit point $+\infty$. The eigenvalues $\{(z_2)_k\}_{k=1}^{\infty}$, $(z_2)_k = (z_2)_k(Z)$ corresponding to these eigenvalues $\{\lambda_k\}_{k=1}^{\infty} \subset [(t_-)^{-1}, +\infty)$ form the Riesz basis in H_0.

References

[1] Shaik VA, Ardekani AM. Squirming in density-stratified fluids. Phys Fluids 2021; 33: (101903). https://doi.org/10.1063/5.0065958

[2] Akiyama S, Waki Y, Okino S, Hanazaki H. Unstable jets generated by a sphere descending in a very strongly stratified fluid. J Fluid Mech. 2019; 867: 26-44. https://doi.org/10.1017/jfm.2019.123

[3] Zhang J, Mercier MJ, Magnaudet J. Core mechanisms of drag enhancement on bodies settling in a stratified fluid. J Fluid Mech 2019; 875: 622-656. https://doi.org/10.1017/jfm.2019.524

[4] More RV, Ardekani FM. Hydrodynamic interactions between swimming microorganisms in a linearly density stratified fluid. Phys Rev E 2021; 103: (013109). https://doi.org/10.1103/PhysRevE.103.013109

[5] Camassa R, Falqui G, Ortenzi G. Hamiltonian aspects of Tthree-layer stratified Ffluids. J Nonlinear Sci. 2021; 31: (70). https://doi.org/10.1007/s00332-021-09726-0

[6] Sreenivasan B, Maurya G. Evolution of forced magnetohydrodynamic waves in a stratified fluid. J Fluid Mech. 2021; 922: (A32). https://doi.org/10.1017/jfm.2021.565

[7] Samodurov AS, Chukharev AM, Kazakov DA. Basic regularities of vertical turbulent exchange in the mixed and stratified layers of the black sea. Phys Oceanogr. 2021; 28(4): 376-391. https://doi.org/10.22449/1573-160X-2021-4-376-391

[8] Zakora DA. Spectral analysis of a viscoelasticity problem. Comput Math Mathemat Phys 2018; 58(11): 1761-1774. https://doi.org/10.1134/S0965542518110179

[9] Forduk KV, Zakora DA. Problem on small motions of a system of bodies filled with ideal fluids under the action of an elastic damping device. Lobachevskii J Math. 2021; 42(5): 889-900. https://doi.org/10.1134/S199508022105005X

[10] Forduk KV, Zakora DA. A problem of normal oscillations of a system of bodies partially filled with ideal fluids under the action of an elastic damping device. Sib Electron Math Rep. 2021; 18(2): 997-1014. https://doi.org/10.33048/semi.2021.18.075

[11] Tsvetkov DO. Oscillations of a liquid partially covered with ice. Lobachevskii J Math. 2021; 42(5): 1078-1093. https://doi.org/10.1134/S199508022105019X

[12] Tsvetkov DO. Oscillations of a stratified liquid partially covered with ice (general case). Math Notes 2020; 107(1): 160-172. https://doi.org/10.1134/S0001434620010150

[13] Tsvetkov DO. Small motions of an ideal stratified fluid with a free surface completely covered with the elastic ice. Sib Elektron Math Izv. 2018; 15: 422-435.

[14] Essaouini H, Bakkali L, Capodanno P. Mathematical study of the small oscillations of two nonmixing fluids, the lower inviscid, the upper viscoelastic, in an open container. J Math Fluid Mech 2017; 19: 645-657. https://doi.org/10.1007/s00021-016-0300-7

[15] Tsvetkov DO. On an initial-boundary value problem which arises in the dynamics of a viscous stratified fluid. Russ Math. 2020; 64: 50-63. https://doi.org/10.3103/S1066369X20080071

[16] Tsvetkov DO. The problem of normal oscillations of a viscous stratified fluid with an elastic membrane. Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Kompyuternye Nauki 2021; 31(2): 311-330. https://doi.org/10.35634/vm210211

[17] Kopachevsky ND, Krein SG. Operator approach to linear problems of hydrodynamics. Vol. 1. Self-adjoint problems for an ideal fluid. Basel, Boston, Berlin, Birkhauser 2001. https://doi.org/10.1007/978-3-0348-8342-9_1

[18] Voytitsky VI. Strong dissipative hydrodynamical systems and the operator pencil of S. Krein. Lobachevskii J Math. 2021; 42(5): 1094-1112. https://doi.org/10.1134/S1995080221050206

[19] Goldstein JA. Semigroups of linear operators and applications. Oxford University Press 1985.