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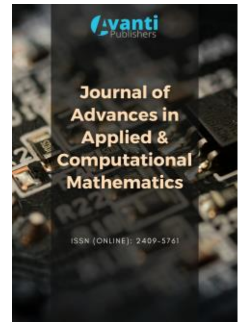

Qualitative Analyses of ψ-Caputo Type Fractional Integrodifferential Equations in Banach Spaces

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ABSTRACT

In this research paper, we develop and extend some qualitative analyses of a class of a nonlinear fractional integro-differential equation involving ψ -Caputo fractional derivative (ψ -CFD) and ψ -Riemann-Liouville fractional integral (ψ -RLFI). The existence and uniqueness theorems are obtained in Banach spaces via an equivalent fractional integral equation with the help of Banach's fixed point theorem (B'sFPT) and Schaefer's fixed point theorem (S'sFPT). An example explaining the main results is also constructed.

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1. Introduction

In this paper, we prove the existence and uniqueness results of a class of IVPs for a nonlinear fractional integrodifferential equation (FIDE):

$$ {}^{C}D_{a^{+}}^{\alpha;\psi}\varpi(t) = F(t,\varpi(t), I_{a^{+}}^{\alpha;\psi}\varpi(t)), \quad t \in J := [a,b], \tag{1.1}$$

$$ \varpi(a) = \varpi_{a}, \varpi_{\psi}^{[k]}(a) = \varpi_{a}^{k}, \quad k = 1, \dots n-1, \tag{1.2}$$

where

i) $n-1 < \alpha < n = [\alpha]+1, n \in \mathbb{N}$, and $\varpi_{a}, \varpi_{a}^{k} \in X$.

ii) ${}^{C}D_{a^{+}}^{\alpha;\psi}$ and $I_{a^{+}}^{\alpha;\psi}$ denote the ψ -CFD and ψ -RLFI of order α, respectively.

iii) $\mathfrak{F}: J \times X \times X \to X$ is continuous.

v) $\varpi \in C^{n-1}(J,X)$ such that the operator ${}^{C}D_{a^{+}}^{\alpha;\psi}$ exists and ${}^{C}D_{a^{+}}^{\alpha;\psi}\varpi \in C(J,X)$.

Fractional calculus (FC) may be taken into consideration as a generalization of classical calculus as there are many definitions for derivatives and integrals of non-integer order. Initially, FC was a purely mathematical concept but currently, its use has spread into many different fields of science and technology such as dynamical systems, physics, biology, engineering, electrochemistry, and bioengineering [1-4] and the references therein. So, in the literature, there are many investigations managing comparative points to different operators, and the most common ones are Riemann-Liouville, Caputo, and Hilfer. Be that as it may, since the operators rely upon the kernel of integration, there are more new types of FDs and integrals that emerge as a result of choosing an alternative kernel that makes the number of definitions broad and more general. For instance, see [5, 6], and the references therein. The utilization of fractional differential equations (FDEs) has arisen as another part of applied mathematics, which has been utilized for developing numerous mathematical models in engineering and science. Indeed FDEs are viewed as model options in contrast to nonlinear DEs and other sorts of equations. On the numerical side and simulation models for this path, we will refer to some recent works in [7-12], whereas the theory of FDEs has been broadly examined by many authors, see [13-21].

The integrodifferential equations (IDEs) emerge in various scientific and engineering specializations. They are predominantly an approximation to PDEs, which epitomize a large part of the continuum phenomena. For points of interest, see [22, 23] and others. Many IDEs can be expressed as fractional IDEs in some Banach spaces [24-29], e.g., Balachandran et al. [24] studied the existence results of Caputo-type quasilinear FIDEs in Banach spaces by using Banach's fixed point theorem. In [30, 31], the authors investigated the existence theorem on abstract FDEs by utilizing semigroup theory and the fixed point technique. Numerous partial differential equations or integrodifferential equations can be communicated as FDEs or integro-FDEs in some Banach spaces [32]. Some recent existence and uniqueness results on FDEs involving generalized FDs have been studied by many authors, see [33-41]. For example, Abdo and Panchal [39] proved the existence, uniqueness, and Ulam-Hyers stability of solutions of ψ -Hilfer FIDEs in weighted spaces via fixed point techniques. Also, Wahash et al. [38] established the existence and uniqueness results for the global solutions of ψ -Caputo singular FDEs by means of Picard's iterative method.

Motivated by the above works, we prove the existence and uniqueness of the solution of a nonlinear FIDE (1.1)–(1.2) involving ψ -CFD by means of B'sFPT and S'sFPT.

The rest of this work is structured as follows. In Section 2, we provide some basic concepts, definitions, and elementary facts that will be useful throughout the paper. The main results are obtained in Section 3. Finally, in Section 4, we provide an illustrative example to justify our results.

2. Preliminaries

Here, we require some essential definitions and properties of ψ -fractional calculus that will be needed in this paper. Let $\alpha > 0$ is a real number, X be a Banach space, $C(J, X)$ be the Banach space of continuous functions $\varkappa(t) \in X$ for $t \in J \to X$ and $\|\varkappa\| = sup_{t\in J} \quad |\varkappa(t)|$. Denote by $L^{1,\psi}(\mathbb{R}^+)$ the set of those Lebesgue integrable functions with respect to ψ, where $\mathbb{R}^+ = [0, \infty)$, and $\|\varkappa\|_{L^{1,\psi}} = \int_a^b \psi'(t)|\varkappa(t)|dt < \infty$, and $\psi \in C^n(J, X)$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in J$,

Definition 2.1 [3] For $\alpha > 0$ and $\varkappa \in L^{1,\psi}(\mathbb{R}^+)$, the left ψ -RLFI is given by

$$I_{a^+}^{\alpha;\psi}\varkappa(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\psi'(s)\varkappa(s)}{(\psi(t) - \psi(s))^{1-\alpha}}ds, \ t > a. \tag{2.1}$$

Definition 2.2 [3] For $n - 1 < \alpha \leq n$, the left ψ -RLFD is defined by

$$D_{a^+}^{\alpha;\psi}\varkappa(t) = D_\psi^n I_{a^+}^{n-\alpha;\psi}\varkappa(t), \tag{2.2}$$

where $D_\psi^n = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n$, $I_{a^+}^{n-\alpha;\psi}$ as in (2.1), and $n = [\alpha] + 1$.

Definition 2.3 [34] For $n - 1 < \alpha \leq n$, the left ψ -CFD is defined by

$${}^C D_{a^+}^{\alpha;\psi}\varkappa(t) = D_{a^+}^{\alpha;\psi}\left[\varkappa(t) - \sum_{k=0}^{n-1}\frac{D_\psi^n\varkappa(a)}{k!}(\psi(t) - \psi(a))^k\right].$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$. Further, for $\alpha \notin \mathbb{N}$ we can write

$$\begin{aligned} {}^C D_{a^+}^{\alpha;\psi}\varkappa(t) &= I_{a^+}^{n-\alpha;\psi}D_\psi^n\varkappa(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{\psi'(s)D_\psi^n\varkappa(s)}{(\psi(t) - \psi(s))^{1-n+\alpha}}ds, \end{aligned} \tag{2.3}$$

and for $\alpha = n \in \mathbb{N}$, one has ${}^C D_{a^+}^{\alpha;\psi}\varkappa(t) = D_\psi^n\varkappa(t)$. In particular, if $n = 1$, then $D_\psi^1\varkappa(t) = \frac{\varkappa'(t)}{\psi'(t)}$, for all $t \in J$.

Note that, if $\psi(t) = t$, then the relations (2.1), (2.2), and (2.3) are reduced to the classical FC (see [3]).

The above definitions have the following properties proved in [3] and [34].

Lemma 2.1 Let $n - 1 < \alpha < n$ and $\varkappa: J \to X$ be a function.

(1) If $\varkappa \in C(J, X)$, then ${}^C D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi}\varkappa(t) = \varkappa(t)$.

(2) If $\varkappa \in C^{n-1}(J, X)$, then $I_{a^+}^{\alpha;\psi}\, {}^C D_{a^+}^{\alpha;\psi}\varkappa(t) = \varkappa(t) - \sum_{k=0}^{n-1}\frac{\varkappa_\psi^{[k]}(a)}{k!}(\psi(t) - \psi(a))^k$.

(3) $I_{a^+}^{\alpha;\psi}\ I_{a^+}^{\beta;\psi}\varkappa(t) = I_{a^+}^{\alpha+\beta;\psi}\ \varkappa(t)$, for $\alpha, \beta > 0$.

(4) $I_{a^+}^{\alpha;\psi}(\cdot)$maps $C(J, X)$ into $C(J, X)$.

(5) $I_{a^+}^{\alpha;\psi}\varkappa(a) = \lim_{t\to a^+} I_{a^+}^{\alpha;\psi}\varkappa(t) = 0$.

Lemma 2.2 [3, 34] Let $n - 1 < \alpha < n$, $n < \beta \in \mathbb{R}$ and let $\psi_{t,a} = \psi(t) - \psi(a)$. Then

(1) $I_{a^+}^{\alpha;\psi}(\psi_{t,a})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\psi_{t,a}\right)^{\alpha+\beta-1}$.

(2) ${}^{C}D_{a^+}^{\alpha;\psi}(\psi_{t,a})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\psi_{t,a}\right)^{\beta-\alpha-1}$.

(3) ${}^{C}D_{a^+}^{\alpha;\psi}(\psi_{t,a})^{k} = 0, \forall k \in \{0,1,\ldots,n-1\}, n \in \mathbb{N}$.

3. Main Results

In this portion, we investigate the existence and uniqueness theorems on (1.1)–(1.2) under B'sFPT and S'sFPT. For convenience, we put $\psi_{t,a} := \psi(t) - \psi(a)$. Firstly, we need the following hypotheses:

$\mathfrak{F}: J \times X^2 \to X$ is continuous and there exists a constant $L > 0$ such that

$$\|\mathfrak{F}(t,\varpi_1,\varpi_2) - \mathfrak{F}(t,v_1,v_2)\| \le L[\|\varpi_1 - v_1\| + \|\varpi_2 - v_2\|], t \in J, \varpi_i, v_i \in X, i = 1,2.$$

There exists an $\rho \in L^{1,\psi}(\mathbb{R}^+)$ such that $\|\mathfrak{F}(t,\varpi,v)\| \le \rho(t)\|\varpi\|$, for all $(t,\varpi,v) \in J \times X^2$.

It is easy to show that problem (1.1)–(1.2) is equivalent to the following FIE:

$$\varpi(t) = \sum_{k=0}^{n-1} \frac{x_a^k}{k!}\left[\psi_{t,a}\right]^k + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi_{t,s})^{\alpha-1}\,\mathfrak{F}(s,\varpi(s), I_{a^+}^{\alpha;\psi}\varpi(s))ds. \tag{3.1}$$

For more details see [39, 40].

By a solution of the problem (1.1)–(1.2), we mean the function ϖ such that the accompanying conditions are fulfilled: (i) $\varpi \in C^{n-1}(J,X)$; (ii) ${}^{C}D_{a^+}^{\alpha;\psi}$ exists and ${}^{C}D_{a^+}^{\alpha;\psi}\varpi \in C(J,X)$, where $0 < \alpha \le 1$; (iii) ϖ satisfies the FIE (3.1).

Now, the first result relying on S'sFPT.

Theorem 3.1 Suppose $\mathfrak{F}: J \times X \times X$ is continuous and (Hy_2) holds. If

$$\frac{\left[\psi_{a+\zeta,a}\right]^{\alpha-1}\|\rho\|_{L^{1,\psi}}}{\Gamma(\alpha)} < 1,\ \zeta > 0,\ a+\zeta \le b, \tag{3.2}$$

then the problem (1.1)-(1.2) has at least one solution on $[a, a+\zeta] \subseteq J$.

Proof. Set the following space

$$\Delta = \{\varpi \in C^{n-1}([a,a+\zeta],X) : {}^{C}D_{a^+}^{\alpha;\psi}\varpi(t) \in C([a,a+\zeta],X)\}. \tag{3.3}$$

In view of (3.1), we define the operator $T: \Delta \to \Delta$ by $T\varpi(t) = \varpi(t)$, i.e.,

$$T\varpi(t) = \sum_{k=0}^{n-1} \frac{x_a^k}{k!}\left[\psi_{t,a}\right]^k + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi_{t,s})^{\alpha-1}\,\mathfrak{F}(s,\varpi(s), I_{a^+}^{\alpha;\psi}\varpi(s))ds. \tag{3.4}$$

Now, we will divide the proof into several steps as follows:

Step 1. T is continuous. Let $\{\varpi_n\}_{n\in\mathbb{N}}$ be a sequence in Δ such that $\varpi_n \to \varpi$ in Δ, as $n \to \infty$. Then for $t \in [a, a+\zeta]$, we have

$$\|T\varpi_n(t) - T\varpi(t)\| = \sup_{t\in[a,a+\zeta]}\left\{\left|I_{a^+}^{\alpha;\psi}\mathfrak{F}(t,\varpi_{\mathrm{n}}(t), I_{a^+}^{\alpha;\psi}\varpi_{\mathrm{n}}(t)) - I_{a^+}^{\alpha;\psi}\mathfrak{F}(t,\varpi(t), I_{a^+}^{\alpha;\psi}\varpi(t))\right|\right\}$$

$$\leq I_{a^+}^{\alpha;\psi}\sup_{t\in[a,a+\zeta]}\left\{\left|\mathfrak{F}(t,\varpi_{\mathrm{n}}(t), I_{a^+}^{\alpha;\psi}\varpi_{\mathrm{n}}(t)) - \mathfrak{F}(t,\varpi(t), I_{a^+}^{\alpha;\psi}\varpi(t))\right|\right\}$$

$$\leq \frac{\left[\psi_{a+\zeta,a}\right]^{\alpha}}{\Gamma(\alpha+1)}\left\|\left(\mathfrak{F}(\cdot,\varpi_n(\cdot), I_{a^+}^{\alpha;\psi}\varpi_n(\cdot)) - \mathfrak{F}(\cdot,\varpi(\cdot), I_{a^+}^{\alpha;\psi}\varpi(\cdot)\right)\right\|.$$

From the continuity of $\mathfrak{F}$, and the Lebesgue dominated convergence theorem, we find $\|T\varpi_n - T\varpi\| \to 0$ as $n \to \infty$. Thus, T is continuous.

Step 2. T is uniformly bounded in Δ.

Here, we prove that, for $r > 0$, there exists some $r^{'} > 0$ such that

$$\forall \varpi \in B_r := \{\varpi \in \Delta : \|\varpi\| \leq r\} : \|T\varpi\| \leq r'.$$

Indeed, let $\varpi \in B_r$, and $t \in [a, a+\zeta]$. Then

$$\|T\varpi\| = \sup_{t\in[a,a+\zeta]}\left\{\left|\sum_{k=0}^{n-1}\frac{|\varpi_a^k|}{k!}\left[\psi_{t,a}\right]^k + I_{a^+}^{\alpha;\psi}\mathfrak{F}(t,\varpi(t), I_{a^+}^{\alpha;\psi}\varpi(t))\right|\right\}$$

$$\leq \sum_{k=0}^{n-1}\frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + I_{a^+}^{\alpha;\psi}\sup_{t\in[a,a+\zeta]}\left|\mathfrak{F}(t,\varpi(t), I_{a^+}^{\alpha;\psi}\varpi(t))\right|$$

$$= \sum_{k=0}^{n-1}\frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + I_{a^+}^{\alpha;\psi}\left\|F(t,\varpi(t), I_{a^+}^{\alpha;\psi}\varpi(t))\right\|$$

$$\leq \sum_{k=0}^{n-1}\frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + I_{a^+}^{\alpha;\psi}\rho(t)\|\varpi\|$$

$$= \sum_{k=0}^{n-1}\frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + I_{a^+}^{\alpha-1;\psi}I_{a^+}^{1;\psi}\rho(t)\|\varpi\|$$

$$\leq \sum_{k=0}^{n-1}\frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + \frac{\left[\psi_{t,a}\right]^{\alpha-1}}{\Gamma(\alpha)}\|\rho\|_{L^{1,\psi}}r$$

$$\leq \sum_{k=0}^{n-1}\frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + \frac{\left[\psi_{a+\zeta,a}\right]^{\alpha-1}}{\Gamma(\alpha)}\|\rho\|_{L^{1,\psi}}r := r'.$$

Thus $\|T\varpi\| \leq r'$. Hence, $\{T\varpi\}$ is uniformly bounded set.

Step3. T is equicontinuous in Δ.

Let $\varpi \in B_r$ such that B_r be bounded set defined as in step 2, and set

$$\underset{(t,\varpi,I_{a^+}^{\alpha;\psi}\varpi)\in a,a+\zeta]\times\mathbb{B}_r\times\mathbb{B}_r}{sup}\left\|\mathfrak{F}(t,\varpi, I_{a^+}^{\alpha;\psi}\varpi)\right\| = \mathfrak{F}_{max}$$

For $t_1, t_2 \in a, a+\zeta]$, with $t_1 < t_2$, we have

$$\|(T\varpi)(t_2)-(T\varpi)(t_1)\| = \left\|\sum_{k=0}^{n-1} \frac{\varpi_a^k}{k!}\left[\psi_{t_2,a}\right]^k + I_{a^+}^{\alpha;\psi}\mathfrak{F}(t_2,\varpi(t_2),I_{a^+}^{\alpha;\psi}\varpi_2(t)) - \sum_{k=0}^{n-1} \frac{\varpi_a^k}{k!}\left[\psi_{t_1,a}\right]^k + I_{a^+}^{\alpha;\psi}\mathfrak{F}(t_1,\varpi(t_1),I_{a^+}^{\alpha;\psi}\varpi_1(t))\right\|$$

$$\leq \sum_{k=0}^{n-1} \frac{|\varpi_a^k|}{k!}\left(\left[\psi_{t_2,a}\right]^k-\left[\psi_{t_1,a}\right]^k\right) + \frac{1}{\Gamma(\alpha)}\int_a^{t_1} \psi'(s)\left[\left[\psi_{t_2,s}\right]^{\alpha-1}-\left[\psi_{t_1,s}\right]^{\alpha-1}\right]\left\|\mathfrak{F}(s,\varpi(s),I_{a^+}^{\alpha;\psi}\varpi(s)\right\|ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \psi'(s)(\psi_{t_2,s})^{\alpha-1}\left\|\mathfrak{F}(s,\varpi(s),I_{a^+}^{\alpha;\psi}\varpi(s)\right\|ds$$

$$\leq \sum_{k=0}^{n-1} \frac{|\varpi_a^k|}{k!}\left(\left[\psi_{t_2,a}\right]^k-\left[\psi_{t_1,a}\right]^k\right) + \frac{\mathfrak{F}_{max}}{\Gamma(\alpha)}\int_a^{t_1}\psi'(s)\left[\left[\psi_{t_1,s}\right]^{\alpha-1}-\left[\psi_{t_2,s}\right]^{\alpha-1}\right]ds + \frac{\mathfrak{F}_{max}}{\Gamma(\alpha)}\int_a^{t_2}\psi'(s)\left(\psi_{t_2,s}\right)^{\alpha-1}ds$$

$$\leq \sum_{k=0}^{n-1} \frac{|\varpi_a^k|}{k!}\left(\left[\psi_{t_2,a}\right]^k-\left[\psi_{t_1,a}\right]^k\right) + \frac{\mathfrak{F}_{max}}{\Gamma(\alpha+1)}\left[\left(\left[\psi_{t_2,a}\right]^\alpha+\left[\psi_{t_2,t_1}\right]^\alpha-\left[\psi_{t_1,a}\right]^\alpha\right)\right]. + \frac{\mathfrak{F}max}{\Gamma(\alpha+1)}\left[\psi_{t_2,t_1}\right]^\alpha$$

$$= \sum_{k=0}^{n-1} \frac{|\varpi_a^k|}{k!}\left([\psi(t_2)-\psi(a)]^k-[\psi(t_1)-\psi(a)]^k\right) + \frac{2\mathfrak{F}_{max}}{\Gamma(\alpha+1)}[\psi(t_2)-\psi(t_1)]^\alpha + \frac{\mathfrak{F}_{max}}{\Gamma(\alpha+1)}[([\psi(t_2)-\psi(a)]^\alpha - [\psi(t_1)-\psi(a)]^\alpha)].$$

Clearly, the R.H.S of the above inequality tends to zero as $t_2 \to t_1$. Therefore $T(\mathbb{B}_r)$ is equicontinuous in Δ. From Arzela-Ascoli theorem, T is completely continuous.

Step 4. We show that the set

$$\Pi = \{\varpi \in \Omega : \varpi = \lambda T\varpi, \text{for some } \lambda \in (0,1)\}$$

is bounded. Let $\varpi \in \Pi$ and $\lambda \in (0,1)$ be such that $\varpi = \lambda T\varpi$. By Step 2, then for all $t \in a, a+\zeta]$, we have

$$|T\varpi(t)| \leq \sum_{k=0}^{n-1} \frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + \frac{\left[\psi_{a+\zeta,a}\right]^{\alpha-1}}{\Gamma(\alpha)}\|\varpi\|\|\rho\|_{L^{1,\psi}}$$

As $\lambda \in (0,1)$ then $\varpi < T\varpi$, and

$$\|\varpi\| < \|T\varpi\|$$

$$\leq \sum_{k=0}^{n-1} \frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k + \frac{\left[\psi_{a+\zeta,a}\right]^{\alpha-1}}{\Gamma(\alpha)}\|\varpi\|\|\rho\|_{L^{1,\psi}}.$$

As per (3.2), we obtain

$$\|\varpi\| \leq \frac{1}{\left(1-\frac{\left[\psi_{a+\zeta,a}\right]^{\alpha-1}}{\Gamma(\alpha)}\|\rho\|_{L^{1,\psi}}\right)}\sum_{k=0}^{n-1} \frac{|\varpi_a^k|}{k!}\left[\psi_{a+\zeta,a}\right]^k,$$

which implies that Π is bounded. An application of S'sFPT shows that there exists at least a fixed point ϖ of T in Δ. Therefore, ϖ is the solution to (1.1)-(1.2) on $[a, a+\zeta] \subseteq J$.

Theorem 3.2. Suppose that (Hy_1) holds. If

$$w := \frac{\left[\psi_{a+\zeta,a}\right]^{\alpha}}{\Gamma(\alpha+1)}L\left(1+\frac{\left[\psi_{a+\zeta,a}\right]^{\alpha}}{\Gamma(\alpha+1)}\right) < 1,\ \zeta \in \mathbb{R}^+,\ a+\zeta \leq b, \tag{3.5}$$

then the problem (1.1)-(1.2) has a unique solution on $[a, a+\zeta] \subseteq J$.

Proof. Consider the set Δ and the operator $T: \Delta \to \Delta$ defined by (3.3) and (3.4), respectively.

We prove that $T(\Delta) \subseteq \Delta$, and T is a contraction mapping. For that, first we consider $\varpi \in C^{n-1}([a, a+\zeta], X)$. It is clear that $T\varpi(t) \in C^{n-1}([a, a+\zeta], X)$. Also, by (3.4), and Lemmas 2.2, 2.1, we have

$$\begin{aligned}{}^C D_{a^+}^{\alpha;\psi}(T\varpi)(t) &= {}^C D_{a^+}^{\alpha;\psi}\sum_{k=0}^{T-1} \frac{\varpi_a^k}{k!}\left[\psi_{t,a}\right]^k + {}^C D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} F(t, \varpi(t), I_{a^+}^{\alpha;\psi}\varpi(t)) \\ &= \mathfrak{F}(t, \varpi(t), I_{a^+}^{\alpha;\psi}\varpi(t)).\end{aligned}$$

Since $\mathfrak{F}$ is continuous on $[a, a+\zeta]$, then ${}^C D_{a^+}^{\alpha;\psi}(T\varpi)(t) \in C([a, a+\zeta], X)$.

Next, let $\varpi_1, \varpi_2 \in \Delta$ and for $t \in a, a+\zeta]$. Then

$$\begin{aligned}\|T\varpi_1(t) - T\varpi_2(t)\| &= \left\|I_{a^+}^{\alpha;\psi}\left[\mathfrak{F}(t, \varpi_1(t), I_{a^+}^{\alpha;\psi}\varpi_1(t)) - \mathfrak{F}(t, \varpi_2(t), I_{a^+}^{\alpha;\psi}\varpi_2(t))\right]\right\| \\ &\leq \frac{\left[\psi_{t,a}\right]^{\alpha}}{\Gamma(\alpha+1)}\left\|\mathfrak{F}(t, \varpi_1(t), I_{a^+}^{\alpha;\psi}\varpi_1(t)) - \mathfrak{F}(t, \varpi_2(t), I_{a^+}^{\alpha;\psi}\varpi_2(t))\right\| \\ &\leq \frac{\left[\psi_{t,a}\right]^{\alpha}}{\Gamma(\alpha+1)}\left[L\|\varpi_1(t) - \varpi_2(t)\| + L\left\|I_{a^+}^{\alpha;\psi}\varpi_1(t) - I_{a^+}^{\alpha;\psi}\varpi_2(t)\right\|\right] \\ &= \frac{\left[\psi_{t,a}\right]^{\alpha}}{\Gamma(\alpha+1)}\left[L\|\varpi_1(t) - \varpi_2(t)\| + L\frac{\left[\psi_{t,a}\right]^{\alpha}}{\Gamma(\alpha+1)}\|\varpi_1(t) - \varpi_2(t)\|\right] \\ &\leq \left[\frac{\left[\psi_{t,a}\right]^{\alpha}}{\Gamma(\alpha+1)}L\left(1+\frac{\left[\psi_{t,a}\right]^{\alpha}}{\Gamma(\alpha+1)}\right)\right]\|\varpi_1(t) - \varpi_2(t)\| \\ &\leq w\|\varpi_1(t) - \varpi_2(t)\|.\end{aligned}$$

Since $w < 1$, T is a contraction. Then by B'sFPT, there exists a unique fixed point $\varpi \in \Delta$ such that $T\varpi(t) = \varpi(t)$. Therefore ϖ is the unique solution to the problem (1.1)-(1.2) on $[a, a+\zeta] \subseteq a, b]$.

4. An Example

Consider the following ψ -CFIDE

$$ {}^{C}D_{a^+}^{\alpha;\psi}\varpi(t)=\frac{e^{-t}\varpi(t)}{(8+e^{t})(1+\varpi(t))}+\frac{1}{9}I_{a^+}^{\alpha;\psi}\varpi(t), \tag{4.1} $$

with initial conditions

$$ \varpi(0)=\varpi_0,\quad \varpi_{\psi}'(0)=\varpi_0^1, \tag{4.2} $$

where $n=2$, $\alpha=\frac{3}{2}$, and $\varpi_0,\varpi_0^1\in X$. Take $X=\mathbb{R}^+$ and $[a,b]=[0,1]$. Set $\mathfrak{F}(t,\varpi,I_{a^+}^{\alpha;\psi}\varpi)=\frac{e^{-t}\varpi}{(9+e^{t})(1+\varpi)}+\frac{1}{9}I_{a^+}^{\alpha;\psi}\varpi$. Let $\varpi,v\in\mathbb{R}^+$ and $t\in 0,1]$. Then

$$ \left\|\mathfrak{F}(t,\varpi,I_{a^+}^{\alpha;\psi}\varpi)-\mathfrak{F}(t,v,I_{a^+}^{\alpha;\psi}v)\right\|\leq\frac{e^{-t}}{(8+e^{t})}\left\|\frac{\varpi-v}{(1+\varpi)(1+v)}\right\|+\frac{1}{9}\left\|I_{a^+}^{\alpha;\psi}\varpi-I_{a^+}^{\alpha;\psi}v\right\| $$

$$ \leq\frac{1}{9}\left[\|\varpi-v\|+\left\|I_{a^+}^{\alpha;\psi}\varpi-I_{a^+}^{\alpha;\psi}v\right\|\right]. $$

Hence, the assumptions (Hy_1) holds with $L=\frac{1}{9}$. We will check that $w<1$. Indeed, we select $\zeta=\frac{1}{2}$, and $\psi(t)=\psi_t=2^t$ for all $t\in 0,1]$. It follows that $w\approx 0.03<1$. Thus, by Theorem 3.2, the problem (4.1)-(4.2) has a unique solution on $[0,\frac{1}{2}]$. Now, we apply Theorem 3.1. For $\varpi\in\mathbb{R}^+$ and $t\in 0,\frac{1}{2}]$, we have

$$ \left\|\mathfrak{F}(t,\varpi,I_{a^+}^{\alpha;\psi}\varpi)\right\|\leq\frac{e^{-t}\|\varpi\|}{(8+e^{t})(1+\|\varpi\|)}+\frac{1}{9}I_{a^+}^{\alpha;\psi}\|\varpi\|\leq\left(\frac{e^{-t}}{(8+e^{t})}+\frac{4\left(\sqrt{2}-1\right)^{\frac{3}{2}}}{27\sqrt{\pi}}\right)\|\varpi\|. $$

Hence, the assumption (Hy_2) holds with $\rho(t)=\frac{e^{-t}}{(8+e^{t})}+\frac{4\left(\sqrt{2}-1\right)^{\frac{3}{2}}}{27\sqrt{\pi}}\in L^{1,\psi}(\mathbb{R}^+)$, due to

$$ \|\rho\|_{L^{1,\psi}}=\int_0^1\quad\psi'(s)\left|\frac{e^{-s}}{(8+e^{s})}+\frac{4\left(\sqrt{2}-1\right)^{\frac{3}{2}}}{27\sqrt{\pi}}\right|ds $$

$$ =\int_0^1\quad 2^s\,log\,2\left|\frac{e^{-s}}{(8+e^{s})}+\frac{4\left(\sqrt{2}-1\right)^{\frac{3}{2}}}{27\sqrt{\pi}}\right|ds $$

$$ =6.1845\times 10^{-2}+\frac{1}{ln\,2}ln\frac{8}{27\sqrt{\pi}}\left(\sqrt{2}-1\right)^{\frac{3}{2}}<\infty. $$

Finally, the condition (3.2) is satisfied, that is $\frac{\left[\psi_{a+\zeta,a}\right]^{\alpha}\|\rho\|_{L^1}}{\Gamma(\alpha+1)}\approx 0.5<1$. By Theorem 3.1, the problem (4.1)-(4.2) has a solution on $[0,\frac{1}{2}]$.

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