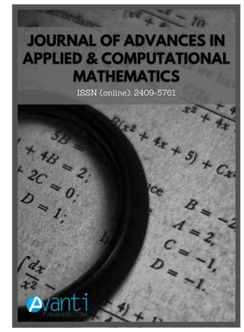
Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761



A Survey on Sharp Oscillation Conditions for Delay Difference Equations

Vasileios Benekas[1,*] and Ardak Kashkynbayev[2,*]

[1]Department.of Mathematics, University of Ioannina, 451 10 Ioannina, Greece.

[2]Department of Mathematics, Nazarbayev University, 010000 Nur-Sultan, Kazakhstan.

ARTICLE INFO

Article Type: Research Article

Keywords:
Oscillation
Several delays
Difference equations
Slowly varying sequence
Non-monotone argument

Timeline:
Received: September 03, 2021
Accepted: November 09, 2021
Published: December 31, 2021

Citation: Benekas V, Kashkynbayev A. A Survey on Sharp Oscillation Conditions for Delay Difference Equations. J Adv App Comput Math. 2021; 8: 117-128.

DOI: https://doi.org/10.15377/2409-5761.2021.08.9

ABSTRACT

Consider the first-order linear difference equation

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n \geq 0, \qquad (1.1)$$

where Δ denotes the forward difference operator, i.e. $\Delta x(n) = x(n+1) - x(n)$, $\{p(n)\}_{n=0}^{\infty}$ is a nonnegative sequence of reals and k is a natural number.

Sharp conditions for the oscillation of all solutions to this equation are presented when the well-known oscillation conditions

$$A := \limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1 \qquad \text{or} \qquad \alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1}$$

are not satisfied. In the case that the sequence $A(n) = \sum_{i=n-k}^{n-1} p(i)$ is *slowly varying at infinity* then under additional assumptions

$$B := \limsup_{n\to\infty} A(n) = \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1}$$

is a sharp condition for the oscillation of all solutions to Eq. (1.1). Analogue sharp oscillation conditions are also presented for the following linear difference equations with constant delays

$\Delta x(n) + \sum_{i=1}^{k} p_i(n)x(n-l_1) = 0$, and variable delays $\Delta x(n) + \sum_{i=1}^{k} p_i(n)x(\tau_i(n)) = 0$,

where $p_i(n): N \to 0,\infty)$, $l_i \in N$ and we assume that there exists a positive integer N such that $0 < l_1 < l_2 < \cdots < l_k \leq N$ holds in the constant delay case, and for variable delays the retarded arguments $\tau_i: N \to Z$ satisfy $n - N \leq \tau_i(n) \leq n - 1$ for all $1 \leq i \leq k$ and $n \in N$.

*Corresponding Author
Email: benekas@gmail.com; ardak.kashkynbayev@nu.edu.kz
Tel: +302651008294, +7(7172)694692

1. Introduction

Consider the first-order linear difference equation

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n \geq 0, \qquad (1.1)$$

where Δ denotes the forward difference operator, i.e. $\Delta x(n) = x(n+1) - x(n)$, $\{p(n)\}_{n=0}^{\infty}$ is a nonnegative sequence of reals and k is a natural number and also the following linear difference equation with several variable delays of the form

$$\Delta x(n) + \sum_{i=1}^{k} p_i(n)x(\tau_i(n)) = 0, \quad (1.2)$$

where $p_i(n): N \to [0,\infty)$, and the retarded arguments $\tau_i: N \to Z$ satisfy $n - N \leq \tau_i(n) \leq n-1$ for all $1 \leq i \leq k$ and $n \in N$.

In this survey, we present the most interesting sharp oscillation conditions for all solutions to the above difference equations.

By a *solution* of the difference equation (1.1), we mean a sequence of real numbers $\{x(n)\}_{n=-k}^{\infty}$ which satisfies Eq.(1.1) for all $n \geq 0$. A solution $\{x(n)\}_{n=-k}^{\infty}$ of the difference equation (1.1) is said to be *oscillatory,* if the terms of the sequence $\{x(n)\}_{n=-k}^{\infty}$ are neither eventually positive nor eventually negative. Otherwise, the solution $\{x(n)\}_{n=-k}^{\infty}$ is said to be *nonoscillatory*.

Similarly, by a solution of the difference equation (1.2), we mean a sequence of real numbers $(x(n))_{n=n_0}^{\infty}$ (with $n_0 \geq -N$), which satisfies equation (1.2), for all integers $n \geq n_0 + N$ and such a solution is said to be *oscillatory,* if $\{x(n)\}_{n=n_0}^{\infty}$ is neither eventually positive nor eventually negative. Otherwise, it is said to be *nonoscillatory*.

2. Oscillations in Linear Difference Equation with Constant Delay

The oscillatory behavior of the solutions to difference equations has gained a great attention in the last three decades. One can find vast literature on this subject such as [1-21] and the references cited therein. For the general theory of difference equations, the reader is referred to the monographs [8, 22-24]. For an overview of the most recent results, we refer to [25], the survey paper [26], and the references therein. In 1981, Domshlak [6] considered the case where $k=1$. In 1989, Erbe and Zhang [7] proved that all solutions of Eq. (1.1) oscillate if

$$\beta := \liminf_{n\to\infty} p(n) > 0 \qquad and \qquad \limsup_{n\to\infty} p(n) > 1 - \beta \qquad (1)$$

$$or \ \liminf_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \qquad (2)$$

$$or \ A := \limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1. \qquad (3)$$

Condition (2) was improved by Ladas, Philos and Sficas [11] by

$$\alpha := \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1}. \qquad (4)$$

This condition turns out to be sharp, since when $p(n)$ is constant, say $p(n) = p$, then it gives

$$p > \frac{k^k}{(k+1)^{k+1}}, \tag{5}$$

which is a *necessary and sufficient condition* [13] for the oscillation of all solutions of Eq. (1.1). Moreover, as it is shown in [7], if

$$sup\, p\,(n) < \frac{k^k}{(k+1)^{k+1}}, \tag{6}$$

then Eq. (1.1) has a nonoscillatory solution.

The conjecture of Ladas in 1990, that Eq. (1.1) has a nonoscillatory solution if

$$\sum_{i=n-k}^{n-1} p(i) \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{for all large } n.$$

turns out to be false as it was shown by a counter-example given in 1994 by Yu, Zhang and Wang [21]. However, in 1999 the "corrected Ladas conjecture"

$$\sum_{i=n-k}^{n} p(i) \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{for all large } n, \tag{7}$$

implies that Eq.(1.1) has a nonoscillatory solution as stated by Tang and Yu [19]. In 2017, Karpuz [9] improved the above result by replacing condition (7) with the weaker condition

$$\sum_{i=n-k}^{n} p(i) \le \left(\frac{k}{k+1}\right)^{k} \quad \text{for all large } n. \tag{8}$$

In the case that the above mentioned conditions (3) and (4) are not satisfied, in 2004 Stavroulakis [16] and in 2006 Chatzarakis and Stavroulakis [4] derived the following sufficient oscillation conditions for Eq. (1.1)

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha^2}{4}, \tag{9}$$

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \alpha^k, \tag{10}$$

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha^2}{2(2-\alpha)}, \tag{11}$$

where α is as in (4) and satisfies $0 < \alpha \le \left(\frac{k}{k+1}\right)^{k+1}$. Also, Chen and Yu [5], derived the following oscillation condition

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{12}$$

In 2001, Shen and Stavroulakis [14], established several oscillation conditions which, in the case of the difference equation with $k = 1$

$$\Delta x(n) + p(n)x(n-1) = 0, \tag{13}$$

reduce to the sufficient oscillation condition

$$\limsup_{n\to\infty} p(n) > \left(\frac{1+\sqrt{1-4\alpha}}{2}\right)^2, \text{ where } 0 \le \alpha \le 1/4. \tag{14}$$

When $\alpha = 1/4$, condition (14) takes the form

$$\limsup_{n\to\infty} p(n) > 1/4 \tag{15}$$

which accordingly to [14] can not be improved in the sense that the lower bound $1/4$ can not be replaced by a smaller number. Indeed, Eq. (13) has a nonoscillatory solution if $\sup p(n) < 1/4$, by condition (6). But, even in the critical state where $\lim_{n\to\infty} p(n) = 1/4$, Eq. (13) can be either oscillatory or nonoscillatory. For example, if $p(n) = \frac{1}{4} + \frac{c}{n^2}$ then Eq. (13) will be oscillatory in case $c > 1/4$ and nonoscillatory in case $c < 1/4$ (the Kneser-like theorem, [6]).

Recall that the difference equation (1.1) is the discrete analogue of the delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \ge t_0, \tag{16}$$

where $p: [t_0, \infty) \to R^+$ is a nonnegative continuous function and τ is a positive constant. In 1972, Ladas, Lakshmikantham and Papadakis [27] proved that all solutions of Eq. (16) oscillate if

$$\mathrm{A} := \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > 1, \tag{17}$$

while in 1982, Koplatadze and Canturija [28] established the following result. If

$$\mathrm{a} := \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}, \tag{18}$$

then all solutions of Eq. (16) oscillate. If

$$\mathrm{A} = \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < \frac{1}{e},$$

or more generally,

$$\int_{t-\tau}^{t} p(s)ds \le \frac{1}{e} \quad \text{for all large } t,$$

then Eq. (16) has a non-oscillatory solution. Observe that in the case that p is a positive constant, the above condition (18) reduces to

$$p\tau > \frac{1}{e}, \tag{19}$$

which is a *necessary and sufficient condition* [8] for all solutions of the delay differential equation

$$x'(t) + px(t-\tau) = 0, \quad p, \tau > 0,$$

to oscillate. Nevertheless, note that $\left(\frac{k}{k+1}\right)^k = \left(\frac{1}{1+\frac{1}{k}}\right)^k \downarrow \frac{1}{e}$ as $k \to \infty$, and therefore conditions (4) and (5) can be interpreted as the discrete analogue of (18) and (19), respectively.

Very recently, Garab et al. [29] essentially improved the above condition (18) by replacing it with

$$\mathrm{A} = \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}, \tag{20}$$

under the additional assumptions that p is a bounded and uniformly continuous function such that

$$\mathrm{a} := \liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > 0, \text{ and } \int_{t-\tau}^{t} p(s)ds \text{ is slowly varying at infinity.}$$

In accordance with [30], a sequence $(a(n))_{n\in N}$ is called *slowly varying (at infinity)*, if

$$\lim_{n\to\infty} \big(a(n+m) - a(n)\big) = 0$$

holds for all $m \in N$. Obviously, this is equivalent to

$$\lim_{n\to\infty} \big(a(n+1) - a(n)\big) = 0.$$

A thorough description of slowly varying functions can be found in the monograph [31], in which, however, a different but related notion of slowly varying functions is treated. For a discussion on the connection of these two notions see [31, Chapter 1].

Pituk came up with the idea to use slowly varying functions for linear delay differential equations with a single constant delay in 2017 [32]. This was generalized in [29, 33, 34] for several variable delays. For difference equations, the first result can be found in [35].

Theorem 1 [35, Theorem 2.1] Assume that $p(n)_{n\in N}$ is nonnegative and bounded such that

$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 0$. Assume also that $A(n) = \sum_{i=n-k}^{n-1} p(i)$ is slowly varying at infinity and

$$B := \limsup_{n\to\infty} A(n) = \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1}. \tag{21}$$

Then all solutions of Eq. (1.1) oscillate.

Example 2 [35, Example 3.1] Consider the difference equation

$$\Delta x(n) + p(n)x(n-2) = 0, \quad n \ge 0, \tag{22}$$

where $p(n) = \frac{1}{9} + \sigma \cos^2(\frac{\pi}{2}\sqrt{n})$ with $\sigma \in (\frac{1}{27}, \frac{1}{9})$. It is clear that $\frac{1}{9} \le p(n) < \frac{2}{9}$ and Eq.(22) is a special case of Eq. (1.1) with $k = 2$.

To show that $p(n)$ is slowly varying at infinity, it suffices to show that $f: [0, +\infty) \to R$ with $f(x) = \frac{1}{9} + \sigma \cos^2(\frac{\pi}{2}\sqrt{x})$ is so. By [29, 32] a continuous function $f: [0, +\infty) \to R$ is slowly varying at infinity if and only if there exists a natural number l_1 and functions $g, h: [l_1, +\infty) \to R$ such that $f(x) = g(x) + h(x)$ on $[l_1, +\infty)$, where g is continuous with $\lim_{x\to+\infty} g(x)$ being a finite number, and h is a continuously differentiable function with $\lim_{x\to+\infty} h'(x) = 0$. It is clear that $f(x)$ satisfies these conditions since $\frac{1}{9}$ is constant and the derivative of $\cos^2(\frac{\pi}{2}\sqrt{x})$ vanishes at infinity. Note that $p(n)$ is slowly varying at infinity which implies that $A(n)$ is also slowly varying.

Furthermore, by the choice of σ, we have

$$B = \limsup_{n\to\infty} A(n) = \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) = \frac{2}{9} + 2\sigma > \left(\frac{2}{3}\right)^3.$$

That is, all conditions of Theorem 1 are satisfied and therefore all solutions of Eq.(22) oscillate.

Next, we show that this conclusion cannot be derived from any of the known results mentioned in Section 1. First we compare our criterion with (1), (2) and (3). Observe that,

$$\beta = \liminf_{n\to\infty} p(n) = 1/9 > 0 \quad \text{and} \quad \limsup_{n\to\infty} p(n) < 2/9 < 1 - 1/9.$$

Moreover,

$$\beta = 1/9 < 4/27$$

and also, by the choice of σ,

$$\limsup_{n\to\infty} \sum_{i=n-2}^{n} p(i) = 3/9 + 3\sigma < 1.$$

Therefore, none of the conditions (1), (2), and (3) is satisfied. Observe also that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-2}^{n-1} p(i) = 2/9 < (2/3)^3$$

and therefore, the condition (4) is not satisfied.

Further, we compare with the conditions (9)-(12). As we have seen, $\alpha = 2/9 < (2/3)^3$ and $B = 2/9 + 2\sigma < 4/9$. Observe that,

$$B < 4/9 < 80/81 = 1 - \alpha^2/4,$$

$$B < 4/9 < 77/81 = 1 - \alpha^2,$$

$$B < 4/9 < \frac{71}{72} = 1 - \frac{\alpha^2}{2(2-\alpha)},$$

and

$$B < 4/9 < \frac{11 + \sqrt{41}}{18} = 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2},$$

and therefore none of the conditions (9), (10), (11), and (12) is satisfied.

3. Oscillations in Linear Difference Equation with Variable Delays

Consider the following linear difference equations with constant delays:

$$\Delta x(n) + \sum_{i=1}^{k} p_i(n) x(n - l_i) = 0 \tag{E}$$

and variable delays:

$$\Delta x(n) + \sum_{i=1}^{k} p_i(n) x(\tau_i(n)) = 0, \tag{1.2}$$

where $p_i(n): N \to [0, \infty)$, $l_i \in N$ and we assume that there exists a positive integer N such that $0 < l_1 < l_2 < \cdots < l_k \leq N$ holds in the constant delay case, and for variable delays the retarded arguments $\tau_i: N \to Z$ satisfy $n - N \leq \tau_i(n) \leq n - 1$ for all $1 \leq i \leq k$ and $n \in N$. Here Δ denotes again the forward difference operator, i.e., $\Delta x(n) = x(n+1) - x(n)$ and N denotes the set of nonnegative integers.

Note that, equation (E) is a special case of equation (1.2) with $\tau_i(n) = n - l_i$ and equation (1.2) can also be written in the form (E) with $k = N$ and different coefficient functions p_i.

Recently, Chatzarakis, Pinelas, and Stavroulakis [36] (corrected version of [37]) obtained the following result for equation (1.2).

Theorem 3 [36, Theorem 2.2] Suppose that the functions $\tau_i(\cdot)$, $1 \leq i \leq k$, are nondecreasing for all $1 \leq i \leq k$. Moreover, assume that

$$\limsup_{n\to\infty} \sum_{i=1}^{k} p_i(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{h=1}^{k} \sum_{j=\tau_i(n)}^{n-1} p_h(j) > \frac{1}{e} \tag{23}$$

hold for all $1 \leq i \leq k$. Then, all solutions of (1.2) oscillate.

As noted in [38], the assumption $\limsup_{n\to\infty} \sum_{i=1}^{k} p_i(n) > 0$ in (23) and of monotonicity of τ_i can be omitted.

Clearly, Theorem 3 works also for (E), that is, for the constant delay.

The following result was essentially obtained by Yan, Meng, and Yan in 2006 [39, Theorem 1], however, as Karpuz and Stavroulakis recently pointed out [25], some corrections were necessary.

Theorem 4 [25, Theorem G] *Assume that*

$$\liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) > 0, \quad \text{holds for some} \quad 1 \leq i \leq k \tag{24}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau_{\max}(n)}^{n-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j - \tau_h(j) + 1}{j - \tau_h(j)} \right)^{j - \tau_h(j) + 1} > 1, \tag{25}$$

where $\tau_{\max}(n) := \max_{1 \leq i \leq k} \tau_i(n)$ for $n \in N$. Then every solution of (1.2) oscillates.

For the case of constant delays the following sharper result was obtained in 1999 by Tang and Yu [19].

Theorem 5 [19, Corollary 4] *Assume that*

$$\liminf_{n\to\infty} \sum_{i=1}^{k} \left(\frac{l_i + 1}{l_i} \right)^{l_i + 1} \sum_{j=n+1}^{n+l_i} p_i(j) > 1.$$

Then every solution of equation (E) is oscillatory.

The concept of slowly varying is used in [38] to improve the above results and also it is assumed that

$$\liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) > 0, \quad \text{holds for all} \quad 1 \le i \le k. \tag{26}$$

The following result in the next theorem improves Theorem 3.

Theorem 6 [38, Theorem 6] Suppose that condition (26) holds and that there exists a positive constant M such that $0 \le p_i(n) \le M$ holds for all $1 \le i \le k$. Assume further that there exists a sequence $(\tau^*(n))_{n\in N}$ such that $\tau_i(n) \le \tau^*(n) \le n-1$ holds for all $1 \le i \le k$ and $n \in N$, and that the function

$$A(n) := \sum_{h=1}^{k} \sum_{j=\tau^*(n)}^{n-1} p_h(j)$$

is slowly varying and the inequality

$$\limsup_{n\to\infty} A(n) = \limsup_{n\to\infty} \sum_{h=1}^{k} \sum_{j=\tau^*(n)}^{n-1} p_h(j) > \frac{1}{e} \tag{27}$$

is fulfilled. Then, all solutions of (1.2) oscillate.

Since equation (E) is a special case of equation (1.2) with $\tau_i(n) = n - l_i$, the following result as an immediate corollary of Theorem 6, is obtained.

Theorem 7 [38, Theorem 7] Suppose that

$$\liminf_{n\to\infty} \sum_{j=n-l_i}^{n-1} p_i(j) > 0 \quad \text{for all} \quad 1 \le i \le k,$$

and there exist a positive constant M and a positive integer $l^* \le l_1$ such that $0 \le p_i(n) \le M$ hold for all $1 \le i \le k$ and the function

$$\sum_{h=1}^{k} \sum_{j=n-l^*}^{n-1} p_h(j)$$

is slowly varying, and moreover, the inequality

$$\limsup_{n\to\infty} \sum_{h=1}^{k} \sum_{j=n-l^*}^{n-1} p_h(j) > \frac{1}{e} \tag{28}$$

is fulfilled. Then all solutions of (E) oscillate.

Analogously to Theorem 6, Theorem 4 gets sharper under some additional assumptions, as follows.

Theorem 8 [38, Theorem 9] Suppose that condition (26) holds and there exists a positive constant M such that $0 \le p_i(n) \le M$ holds for all $1 \le i \le k$. Assume further that there exists a sequence $(\tau^*(n))_{n\in N}$ such that $\tau_i(n) \le \tau^*(n) \le n-1$ holds for all $1 \le i \le k$ and $n \in N$, and that the function

$$A(n) \sum_{j=\tau^*(n)}^{n-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j-\tau_h(j)+1}{j-\tau_h(j)}\right)^{j-\tau_h(j)+1} \tag{29}$$

is slowly varying and the inequality

$$\limsup_{n\to\infty} \sum_{j=\tau^*(n)}^{n-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j-\tau_h(j)+1}{j-\tau_h(j)}\right)^{j-\tau_h(j)+1} > 1$$

is fulfilled. Then all solutions of equation (1.2) are oscillatory.

Similarly Theorem 5 is improved by the following result.

Theorem 9 [38, Theorem 10] Suppose that

$$\liminf_{n\to\infty} \sum_{j=n-l_i}^{n-1} p_i(j) > 0 \quad \text{for all} \quad 1 \le i \le k,$$

and there exists $M > 0$ such that $0 \le p_i(n) \le M$ hold for all $1 \le i \le k$. Furthermore, assume that the function

$$\sum_{i=1}^{k} \left(\frac{l_i+1}{l_i}\right)^{l_i+1} \sum_{j=n+1}^{n+l_i} p_i(j)$$

is slowly varying, and the inequality

$$\limsup_{n\to\infty} \sum_{i=1}^{k} \left(\frac{l_i+1}{l_i}\right)^{l_i+1} \sum_{j=n+1}^{n+l_i} p_i(j) > 1$$

holds. Then every solution of equation (E) is oscillatory.

The last two Theorems improve Theorem 1 because it is a special case with $k = 1$ and $\tau^*(n) = \tau_1(n)$.

The following Example compares Theorems 6 and 3.

Example 10 [38, Example 3.1] Let us consider equation (1.2) with $k = 2$ and $N = 3$. We suppose that $\tau_i(n) \in \{n-2, n-3\}$ for all $n \in N$ and $i = 1,2$, but let's not make further assumptions on τ_1 and τ_2 for now. Furthermore, let the coefficient functions be defined by

$$p_1(n) = c_1 + d_1 \cos^2(\sqrt{n} \cdot \pi/2) + (-1)^n \varepsilon, \quad \text{and} \quad p_2(n) = c_2$$

for all $n \in N$, where c_1, c_2, d_1 and ε are positive parameters with $\varepsilon < c_1$. Then, by choosing $\tau^*(n) = n-2$ for all $n \in N$, it is elementary to show that the sequence

$$A(n) \sum_{h=1}^{k} \sum_{j=\tau^*(n)}^{n-1} p_h(j) = 2c_1 + 2c_2 + d_1 \sum_{j=1}^{2} \cos^2(\sqrt{n-j} \cdot \pi/2)$$

is slowly varying at infinity and that

$$\limsup_{n\to\infty} A(n) = 2(c_1 + c_2 + d_1)$$

(consider the subsequence $\delta(n) = 4n^2 + 1$). On the other hand, the coefficient functions are clearly nonnegative and bounded, and moreover, condition (26) also holds. Hence, the application of Theorem 6 implies that all solutions oscillate, whenever

$$2(c_1 + c_2 + d_1) > \frac{1}{e} \tag{30}$$

is fulfilled.

On the other hand, if, for example, $\tau_1(n) = n - 2$ and $\tau_2(n) = n - 3$ for all $n \in N$, then it is not hard to check that Theorem 3 can only guarantee oscillation of all solutions if

$$\liminf_{n\to\infty} \sum_{h=1}^{k} \sum_{j=n-2}^{n-1} p_h(j) = 2c_1 + 2c_2 + d_1 \liminf_{n\to\infty} \sum_{j=1}^{2} \cos^2(\sqrt{n-j}\cdot \pi/2) = 2(c_1 + c_2) > \frac{1}{e},$$

which is clearly more restrictive than condition (30).

In the constant (single) delay case with $\tau_1(n) = \tau_2(n) \equiv n - 3$, we should choose $\tau^*(n) = n - 2$ for these coefficient functions. This is because with the choice of $\tau^*(n) \equiv n - 3$, we would get

$$A(n) = 3c_1 + 3c_2 + (-1)^{n-1}\varepsilon + d_1 \sum_{j=1}^{3} \cos^2(\sqrt{n-j}\cdot \pi/2),$$

resulting in $\lim_{n\to\infty} | A(n+1) - A(n) | = 2\varepsilon \neq 0$, meaning that $A(n)$ is not slowly varying and one could not apply Theorem 6 with this choice of τ^*.

Finally, an application of Theorem 9 is given.

Example 11 [38, Example 3.2] Consider the constant delay equation (27) with $l_1 = 1$ and $l_2 = 2$ and coefficient functions

$$p_1(n) = c_1 + d_1 \cos^2(\sqrt{n}\cdot \pi/2), \quad \text{and} \quad p_2(n) = c_2,$$

with positive constants c_1, c_2 and d_1. The coefficients are evidently nonnegative and bounded, moreover, condition (26) is satisfied.

Then the function

$$\sum_{i=1}^{k} \left(\frac{l_i + 1}{l_i}\right)^{l_i+1} \sum_{j=n+1}^{n+l_i} p_i(j) = 4\left(c_1 + d_1 \cos^2(\sqrt{n+1}\cdot \pi/2)\right) + \left(\frac{3}{2}\right)^3 2c_2$$

is slowly varying at infinity, so Theorem 9 can be applied to obtain that all solutions are oscillatory, provided

$$4(c_1 + d_1 \limsup_{n\to\infty} \cos^2(\sqrt{n+1}\cdot \pi/2)) + \frac{27}{4} c_2 = 4(c_1 + d_1) + \frac{27}{4} c_2 > 1.$$

On the other hand, Theorem 4 can guarantee oscillation of all solutions only if the stronger condition $4c_1 + \frac{27}{4} c_2 > 1$ is satisfied.

Summary In this paper, we present sharp oscillation conditions for difference equations with one and several delays. The conditions essentially improve the related existing ones from $liminf$ to $limsup$. The same problem can be considered for non-linear and also for higher order difference equations.

Acknowledgments

The authors would like to thank the reviewers for useful suggestions which improved the presentation of the paper.

References

[1] Baštinec J, Berezansky L, Diblík J, Šmarda Z. A Final Result on the Oscillation of Solutions of the Linear Discrete Delayed Equation with a Positive Coefficient. Abstr Appl Anal 2011; Article ID 586328 (2011) https://doi.org/10.1155/2011/586328

[2] Baštinec J, Diblík J, Šmarda Z. Oscillation of Solutions of a Linear Second-Order Discrete-Delayed Equation. Adv Differ Equ 2010; Article ID 693867 (2010) https://doi.org/10.1186/1687-1847-2010-693867

[3] Braverman E, Karpuz B. On oscillation of differential and difference equations with non-monotone delays. Appl Math Comput 2011; 218: 3880-3887https://doi.org/10.1016/j.amc.2011.09.035

[4] Chatzarakis GE, Stavroulakis IP. Oscillations of first order linear delay difference equations. Austral J Math Anal Appl 2006; 3(1): Art.14, 1-11.

[5] Chen MP, Yu YS. Oscillations of delay difference equations with variable coefficients. Proc. First Intl. Conference on Difference Equations, (Edited by S.N. Elaydi et al), Gordon and Breach, 1995; 105-114.

[6] Domshlak Y. Discrete version of Sturmian Comparison Theorem for non-symmetric equations. Dokl Azerb Acad Sci 1981; 37: 12-15 (in Russian).

[7] Erbe LH, Zhang BG. Oscillation of discrete analogues of delay equations. Differential Integral Equations 1989; 2: 300-309.

[8] Györi I, Ladas G. Oscillation Theory of Delay Differential Equations with Applications. Clarendon Press, Oxford 1991.

[9] Karpuz B. Sharp oscillation and nonoscillation tests for linear difference equations. J Difference Equ Appl 2017; 23(12): 1229-1242 https://doi.org/10.1080/10236198.2017.1379515

[10] Ladas G. Recent developments in the oscillation of delay difference equations. In International Conference on Differential Equations, Stability and Control, Dekker, New York 1990.

[11] Ladas G, Philos ChG, Sficas YG. Sharp conditions for the oscillation of delay difference equations. J Appl Math Simulation, 1989; 2: 101-112 https://doi.org/10.1155/S1048953389000080

[12] Moremedi GM, Stavroulakis IP. Oscillation conditions for difference equations with a monotone or non-monotone argument. Discr Dyn Nat Soc 2018; Article ID 9416319 (2018) https://doi.org/10.1155/2018/9416319

[13] Moremedi GM, Stavroulakis IP, Zhunussova ZK. Necessary and Sufficient Conditions for Oscillations of Functional Differential Equations. J Math Mech Comput Sci 2018; 99(3): 12-23 https://doi.org/10.26577/JMMCS-2018-3-528

[14] Shen JH, Stavroulakis IP. Oscillation criteria for delay difference equations. Electron J. Diff Eqns 2001; 10: 1-15.

[15] Stavroulakis IP. Oscillations of delay difference equations. Comput Math Applic 1995; 29: 83-88 https://doi.org/10.1016/0898-1221(95)00020-Y

[16] Stavroulakis IP. Oscillation criteria for first order delay difference equations Mediterr J Math 2004; 1: 231-240 https://doi.org/10.1007/s00009-004-0013-7

[17] Stavroulakis IP. A Survey on the Oscillation of Difference Equations with Constant Delays. J Math Mech Comput Sci 2019; 102(2): 3-11 https://doi.org/10.26577/JMMCS-2019-2-21

[18] Tang XH, Deng YB. Oscillation of difference equations with several delays. Hunan Daxue Xuebao 1998; 25(2): 1-3.

[19] Tang XH, Yu JS. Oscillation of delay difference equation. Comput Math Appl 1999; 37(7): 11-20. https://doi.org/10.1016/S0898-1221(99)00083-8

[20] Tang XH, Zhang RY. New oscillation criteria for delay difference equations. Comput Math Applic 2001; 42(10-11): 1319-1330. https://doi.org/10.1016/S0898-1221(01)00243-7

[21] Yu JS, Zhang BG, Wang ZC. Oscillation of delay difference equations. Appl Anal 1994; 53: 117-124 https://doi.org/10.1080/00036819408840248

[22] Agarwal P. Difference Equations and Inequalities. Marcel Dekker, 1992.

[23] Agarwal RP, Bohner M, Grace SR, O'Regan D. Discrete Oscillation Theory. Hindawi Publishing Corporation, New York 2005; https://doi.org/10.1155/9789775945198

[24] Lakshmikantham V, Trigiante D. Theory of difference equations: Numerical methods and applications. Mathematics in Science and Engineering, 181, Academic Press, Boston, MA 1998.

[25] Karpuz B, Stavroulakis IP. Oscillation and nonoscillation of difference equations with several delays. Mediterr J Math 2021; 18(1): Paper 15 pp. https://doi.org/10.1007/s00009-020-01617-0

[26] Stavroulakis IP. Oscillations of delay and difference equations with variable coefficients and arguments, in: Differential and difference equations with applications, Springer Proc Math Stat 2016; 164: pp. 169-189. https://doi.org/10.1007/978-3-319-32857-7_17

[27] Ladas G, Laskhmikantham V, Papadakis JS. Oscillations of higher-order retarded differential equations generated by retarded arguments. Delay and Functional Differential Equations and Their Applications, Academic Press, New York 1972; 219-231. https://doi.org/10.1016/B978-0-12-627250-5.50013-7

[28] Koplatadze RG, Chanturija TA. On the oscillatory and monotonic solutions of first order differential equations with deviating arguments, Differentsial'nye Uravneniya 1982; 18: 1463-1465.

[29] Garab Á, Pituk M, Stavroulakis IP. A sharp oscillation criterion for a linear delay differential equation. Appl Math Letters 2019; 93: 58-65 https://doi.org/10.1016/j.aml.2019.01.042

[30] Ash JM, Erdös P, Rubel LA. Very slowly varying functions, Aequat. Math. 1974; 10: 1-9. https://doi.org/10.1007/BF01834775

[31] Seneta E. Regularly varying functions, Lecture Notes in Mathematics, 1976; 508, Springer-Verlag, Berlin-New York, https://doi.org/10.1007/BFb0079658

[32] Pituk M. Oscillation of a linear delay differential equation with slowly varying coefficient. Appl Math Letters 2017; 73: 29-36. https://doi.org/10.1016/j.aml.2017.04.019

[33] Garab Á. A sharp oscillation criterion for a linear differential equation with variable delay. Symmetry. 2019; 11(11): Article ID 1332. https://doi.org/10.3390/sym11111332

[34] Garab Á, Stavroulakis IP. Oscillation criteria for first order linear delay differential equations with several variable delays. Appl Math Lett 2020; 106: Article ID 106366 https://doi.org/10.1016/j.aml.2020.106366

[35] Benekas V, Kashkynbayev A, Stavroulakis IP. A sharp oscillation criterion for a difference equation with constant delay. Adv Differ Equ 2020; 566 (2020). https://doi.org/10.1186/s13662-020-03016-x

[36] Chatzarakis GE, Pinelas S, Stavroulakis IP. Correction to: Oscillations of difference equations with several deviated arguments. Aequationes Math 2018; 92: 1195-1200. https://doi.org/10.1007/s00010-018-0581-4

[37] Chatzarakis GE, Pinelas S, Stavroulakis IP. Oscillations of difference equations with several deviated arguments. Aequationes Math 2014; 88: 105-123. https://doi.org/10.1007/s00010-013-0238-2

[38] Benekas V, Garab Á, Kashkynbayev A, Stavroulakis IP. Oscillation criteria for linear difference equations with several variable delays. Opuscula Math 2021; 41(5): 613-627 (2021). https://doi.org/10.7494/OpMath.2021.41.5.613

[39] Yan W, Meng Q, Yan JR. Oscillation criteria for difference equation of variable delays. Dyn Contin Discret Impuls Syst Ser A Math Anal 2006; 13A(Part 2): 641-647.