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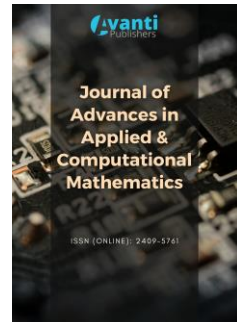


# A Conjecture Congenetic with Fermat's Last Theorem

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## ABSTRACT

We propose the conjecture that for any positive integers $r$ and $n$ with $n > 2$, there do not exist $2r + 1$ consecutive positive integers in natural order such that the sum of $n$-th powers of the first $r + 1$ integers equals the sum of $n$-th powers of the subsequent $r$ integers, i.e., there are no positive integers $r$, $m$ and $n$, where $r < m$ and $n > 2$, satisfying $(m-r)^n + (m-r+1)^n + \cdots + m^n = (m+1)^n + (m+2)^n + \cdots + (m+r)^n$. We prove that the conjecture is true for the cases $n = 3$ and $n = 4$. We also verified by using Mathematica that the conjecture is true for the cases $3 \le n \le 10$ and $m \le 5000$.

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# 1. The Proposed Conjecture

Fermat's last theorem [1-3], also known as Fermat's conjecture, is the statement that there are no positive integers $a,b,c,n$ with $n>2$ such that

$$a^n + b^n = c^n. \tag{1}$$

The statement was written in 1637 by the famous French mathematician Pierre de Fermat in his copy of the Arithmetica by Diophantus of Alexandria [4]. In history, Fermat's conjecture was called a theorem because it was believed that the conjecture was true, although a complete proof was not given yet. Many scholars, including some famous mathematicians, devoted themselves to proving Fermat's conjecture, but gave only a partial solution to the problem for designated values of $n$ [5]. It was not until the end of the last century that A. J. Wiles presented complete proof by using new and advanced appliances in algebraic geometry and number theory [6, 7]. In this paper, we propose a conjecture as follows.

**Conjecture** For any positive integers $r$ and $n$ with $n>2$, there do not exist $2r+1$ consecutive positive integers in natural order such that the sum of $n$-th powers of the first $r+1$ integers equals the sum of $n$-th powers of the subsequent $r$ integers. In other words, there are no positive integers $r,m,n$ with $r<m$ and $n>2$ such that

$$(m-r)^n + (m-r+1)^n + \ldots + m^n = (m+1)^n + (m+2)^n + \ldots + (m+r)^n. \tag{2}$$

We can say that our conjecture is congenetic with Fermat's last theorem. Fermat wrote down his conjecture just beside the content about indeterminate equations, alias Diophantine equations [8, 9], in the form of Eq. (1) in his copy of the Arithmetica. So definitely Fermat had made careful consideration for this type of indeterminate equations, especially for the Pythagorean triple $(a,b,c)$ satisfying the indeterminate equation $a^2+b^2=c^2$ [10-12]. In history, the earliest discovered Pythagorean triple is (3, 4, 5), which was found on papyrus of 2600 BC in ancient Egypt [12, 13]. From the equation $3^2+4^2=5^2$ to Eq. (1) in Fermat's last theorem, besides the raising of power exponent, three consecutive positive integers 3, 4, 5 were generalized to three arbitrary positive integers. Whereas in our conjecture, besides the raising of power exponent, three consecutive positive integers 3, 4, 5 were generalized to any $2r+1$ consecutive positive integers.

In next section, we prove that for the case $n=1$ and the case $n=2$, for any specified positive integer $r$, Eq. (2) has a unique solution for $m$, while for the case $n=3$ and the case $n=4$, our conjecture is true.

# 2. The Cases for $n=1,2,3,4$

First, we give equivalent forms to Eq. (2). In a compact form, Eq. (2) is

$$\sum_{p=0}^{r}(m-p)^n = \sum_{p=1}^{r}(m+p)^n. \tag{3}$$

Expanding by using the binomial formula yields

$$m^n + \sum_{p=1}^{r}\sum_{j=0}^{n} C_n^j m^{n-j}(-p)^j = \sum_{p=1}^{r}\sum_{j=0}^{n} C_n^j m^{n-j} p^j, \tag{4}$$

where $C_n^j = \dfrac{n!}{j!(n-j)!}$ is the combination coefficient.

If $n$ is odd, then Eq. (4) is simplified as

$$m^n = 2\sum_{p=1}^{r}\left(C_n^1 m^{n-1} p + C_n^3 m^{n-3} p^3 + \ldots + C_n^n p^n\right). \tag{5}$$

If $n$ is even, then Eq. (4) is simplified as

$$m^n = 2\sum_{p=1}^{r}\left(C_n^1 m^{n-1} p + C_n^3 m^{n-3} p^3 + \ldots + C_n^{n-1} m p^{n-1}\right),$$

i.e.,

$$m^{n-1} = 2\sum_{p=1}^{r}\left(C_n^1 m^{n-2} p + C_n^3 m^{n-4} p^3 + \ldots + C_n^{n-1} p^{n-1}\right). \tag{6}$$

Next, we give the results for the cases of $n = 1$ and $n = 2$.

**Theorem 1** If $n = 1,$ then for any $r \in N^+$, Eq. (2) has a unique solution $m = r^2 + r$. If $n = 2,$ then for any $r \in N^+$, Eq. (2) has a unique solution $m = 2r^2 + 2r$.

*Proof.* If $n = 1$, then the unique solution follows from Eq. (5), $m = 2\sum_{p=1}^{r} p = r^2 + r.$ If $n = 2$, then we have the unique solution from Eq. (6), $m = 4\sum_{p=1}^{r} p = 2r^2 + 2r$. The proof is completed. W

From the above results, for $n = 1$ and any $r \in N^+$, the $2r + 1$ consecutive positive integers determined by Eq. (2) are

$$r^2, r^2 + 1, r^2 + 2, \ldots, r^2 + 2r.$$

For $n = 2$ and any $r \in N^+$, the $2r + 1$ consecutive positive integers determined by Eq. (2) are

$$2r^2 + r, 2r^2 + r + 1, \ldots, 2r^2 + 3r.$$

Now we prove our conjecture for the cases of $n = 3$ and $n = 4.$

**Theorem 2** For the case $n = 3$, our conjecture is true, i.e., Eq. (2) does not have a solution for $r, m \in N^+$ and $r < m$.

*Proof.* For the considered case, Eq. (5) becomes

$$m^3 = 2\sum_{p=1}^{r}(3m^2 p + p^3). \tag{7}$$

Applying the formula

$$\sum_{p=1}^{r} p^3 = \frac{r^2(r+1)^2}{4}, \tag{8}$$

we have

$$m^3 = 3m^2 r(r+1) + \frac{1}{2} r^2 (r+1)^2. \tag{9}$$

Due to $r(r+1)$ being even, $r(r+1) = 2q$ for some $q \in N^+$, $m$ cannot be odd. Suppose $m = 2l$ for some $l \in N^+$. Inserting these expressions into Eq. (9) leads to

$$q^2 + 12l^2 q - 4l^3 = 0. \tag{10}$$

Solving for $q$ yields

$$q = -6l^2 + 2l\sqrt{9l^2 + l}. \tag{11}$$

Suppose $9l^2 + l$ is a square number such that $9l^2 + l = u^2$ for some $u \in N^+$. Then it follows that $(18l+1)^2 = (6u)^2 + 1$, i.e., we find two square numbers and their difference is 1, which is a contradiction. So $9l^2 + l$ cannot be a square number. Thus $q$ cannot be a positive integer. This is contradictory to the hypothesis of $q$. Therefore, Eq. (2) does not have positive integer solutions for $r, m \in N^+$ and $r < m$. The proof is completed. W

**Theorem 3** For the case $n = 4$, our conjecture is true, i.e., Eq. (2) does not have a solution for $r, m \in N^+$ and $r < m$.

*Proof.* For this case, Eq. (6) becomes

$$m^3 = 2\sum_{p=1}^{r} (4m^2 p + 4p^3). \tag{12}$$

Applying the formula (8), we have

$$m^3 = 4m^2 r(r+1) + 2r^2 (r+1)^2. \tag{13}$$

Due to $r(r+1)$ being even, $m$ can only be even. Suppose $m = 2l$ for some $l \in N^+$. So Eq. (13) becomes

$$r^2 (r+1)^2 + 8l^2 r(r+1) - 4l^3 = 0. \tag{14}$$

Solving for $r(r+1)$ yields

$$r(r+1) = -4l^2 + 2l\sqrt{4l^2 + l}. \tag{15}$$

Suppose $4l^2 + l$ is a square number such that $4l^2 + l = u^2$ for some $u \in N^+$. Then it follows that $(8l+1)^2 = (4u)^2 + 1$, which is a contradiction. So $4l^2 + l$ cannot be a square number. Thus $r(r+1)$ is not a positive integer. This contradiction shows that Eq. (2) does not admit positive integer solutions for $r, m \in N^+$ and $r < m$. The proof is completed. W

For the case of $n$ more than 4, we cannot prove or disprove the proposed conjecture. We indicate that the restriction "$2r+1$ consecutive" in our conjecture cannot be dropped since there are the examples $1^3 + 5^3 + 9^3 = 7^3 + 8^3$ and $31^3 + 35^3 + 37^3 = 39^3 + 40^3$.

## 3. Test Using Mathematica

We use the command 'Reduce' in Mathematica 11 to test our results [14, 15]. Running the code

n = 1; Reduce[Sum[(m - p)^n, {p, 0, r}] == Sum[(m + p)^n, {p, 1, r}] && 1 <= r < m < 100, {r, m}, Integers]

outputs

(r == 1 && m == 2) || (r == 2 && m == 6) || (r == 3 && m == 12) || (r == 4 && m == 20) || (r == 5 && m == 30) || (r == 6 && m == 42) || (r == 7 && m == 56) || (r == 8 && m == 72) || (r == 9 && m == 90).

Running the code

n = 2; Reduce[Sum[(m - p)^n, {p, 0, r}] == Sum[(m + p)^n, {p, 1, r}] && 1 <= r < m < 100, {r, m}, Integers]

outputs

(r == 1 && m == 4) || (r == 2 && m == 12) || (r == 3 && m == 24) || (r == 4 && m == 40) || (r == 5 && m == 60) || (r == 6 && m == 84).

So we verified Theorem 1 with the limitation $m < 100$. By running the following code,

n=.; Reduce[Sum[(m - p)^n, {p, 0, r}] == Sum[(m + p)^n, {p, 1, r}] && 3 <= n <= 10 && 1 <= r < m <= 5000, {n, r, m}, Integers]

Mathematica showed that there is no solution. Thus we verified that our conjecture is true for the cases $3 \le n \le 10$ and $m \le 5000$.

## 4. Conclusion

Eq. (2) is a natural generalization of the equation satisfied by the Pythagorean triple (3, 4, 5). Such generalization is somewhat similar to that from the Pythagorean triple to Eq. (1). Eq. (2) is an indeterminate equation in $n, r, m \in N^+$, where $r < m$. We proved that for the cases $n = 1$ and $n = 2$, Eq. (2) has solutions and for any $r \in N^+$ there is a unique $m$ satisfying Eq. (2), while for the cases $n = 3$ and $n = 4$, Eq. (2) does not have a solution for $r, m \in N^+$ and $r < m$. These show some analogy with Eq. (1). Hence, we propose the conjecture that there are no positive integers $n, r, m$, where $n > 2$ and $r < m$, satisfying Eq. (2). We proved that the conjecture is true for the cases $n = 3$ and $n = 4$. We also verified by using Mathematica that the conjecture is true for the cases $3 \le n \le 10$ and $m \le 5000$.

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