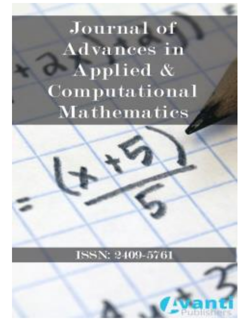
Published by Avanti Publishers

Journal of Advances in Applied & Computational Mathematics

ISSN (online): 2409-5761



Analysis of Functional and Neutral Differential Equations via Lyapunov Functionals

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ARTICLE INFO

Article Type: Research Article

Academic Editor: Hüseyin Budak

Keywords:
Neutral
Infinite delay
Boundedness
Functional delay
Exponential stability
Lyapunov functionals

Timeline:
Received: November 08, 2023
Accepted: December 12, 2023
Published: December 29, 2023

Citation: Raffoul YN. Analysis of functional and neutral differential equations via lyapunov functionals. J Adv App Comput Math. 2023; 10: 120-136.

DOI: https://doi.org/10.15377/2409-5761.2023.10.11

ABSTRACT

We employ Lyapunov types functions and functionals and obtain sufficient conditions that guarantee the boundedness and the exponential decay of solutions, stability and exponential stability of the zero solution in nonlinear delay and neutral differential systems. The theory is illustrated with several examples.

2000 Mathematics Subject Classification: Primary 39A13, 39A23; Secondary 34K42.

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1. Introduction

Lyapunov functionals play a crucial role in the analysis of dynamic systems, including delay systems. The application of Lyapunov functionals in delay systems is both fruitful and challenging for several reasons. Many real-world systems are nonlinear and complex. Lyapunov functionals provide a powerful tool for analyzing the stability of nonlinear systems, but in the context of delay systems, the combination of nonlinearity and delay introduces additional challenges in constructing Lyapunov functionals that accurately capture the system's dynamics. Employing Lyapunov functionals in delay systems is fruitful because it provides a systematic approach to analyze stability and understand the impact of delays on system behavior. However, challenges arise in constructing suitable Lyapunov functionals due to the unique characteristics of delay systems, including non-local dependencies, nonlinearity, and computational complexity. Researchers continue to explore and develop techniques to address these challenges and enhance the applicability of Lyapunov functionals in the analysis of delay systems. In this report, we intend to introduce different types of Lyapunov functionals that are suited for different types of delay systems. We intend to prove general theorems that require the existence of such Lyapunov functionals in arriving at our specific results. Examples are provided throughout this paper.

Let R^n denote the n -dimensional Euclidean vector space; R^+ denote the set of all non-negative real numbers; $|\phi|$ denote the Euclidean norm of a vector $x \in R^n$. For $h > 0$, define the set $B_h(0, H)$ to be the space of all continuously differentiable functions $\phi: [-h, 0] \to R^n$ such that $|\phi| \leq H$, where $|\phi| = \max_{t\in[-h,0]} ||\phi(t)||$. This paper considers the boundedness of solutions and stability of the zero solution of the functional delay differential system

$$x' = f(t, x_t,), \quad t \geq 0, \tag{1.1}$$
$$x(s) = \phi(s), \quad s \in [-h, 0], \quad h > 0$$

where $x \in R^n$, $f: R^+ \times R^n \times R^n \to R^n$ is a given nonlinear continuous function in t and x, where $t \in R^+$. The notation x_t means that $x_t = x(t+\tau), \tau \in [-h, 0]$ as long as $x(t+\tau)$ is defined. Thus, x_t is a function on an interval $[-h, 0]$ into R^n. Let $\phi \in B_h(0, H)$. We say that $x(t) \equiv x(t, t_0, \phi)$ is a solution of (1.1) if $x(t)$ satisfies (1.1) for $t \geq t_0$ and $x_{t_0} = x(t_0 + s) = \phi(s)$, $s \in [-h, 0]$. Our interest in studying the boundedness of solutions of (1.1), beside other topics arise from the fact that in the study of the stability of the zero solution when $f(t, 0) = 0$, one may have to assume the existence of solutions for all positive time. For more on the stability, we refer the interested reader to [1, 2]. In [3], the author obtained by using positive definite Lyapunov functionals interesting results regarding the boundedness of solutions of systems that are similar to (1.1). For more results regarding the stability of the zero solution or boundedness of solutions of (1.1), we refer the interested reader to [4-7]. Inspired by the work in [2, 8], in this investigation, we establish sufficient conditions yielding that all solutions of (1.1) are bounded and obtain results regarding the stability of the zero solution when $f(t, 0) = 0$. We achieve this by assuming the existence of a Lyapunov function that is bounded below and above and that its derivative along the trajectories of (1.1) to be bounded by a negative definite function, plus a positive constant. Our motivation stem from the fact that finding a suitable Lyapunov function or functional presents big challenges when the system has delay terms or of neutral type.

System (1.1) encompasses linearly delayed or retarded models. Linear retarded differential equations are a specific type of ordinary differential equations that might include a time delay in the derivative term. These equations are used in various fields of science and engineering to model systems where the response of a variable depends on its past values with a delay. Linear retarded differential equations have applications in control systems and chemical processes, which often involve reaction kinetics and transport phenomena with time delays. The application of linear systems with delay is also found in the fields of environmental sciences, mechanical and structural engineering, economics, population dynamics, and heat transfer. These are just a few examples of the many applications of linearly retarded differential equations in various fields. They provide a powerful tool for understanding and predicting the behavior of systems with inherent time delays. Typically, research on linear systems of differential equations with delays involves developing mathematical models, analyzing system behavior, and designing control strategies to achieve desired system performance. Engineers in various disciplines collaborate with mathematicians to address these complex problems and develop practical solutions. Most engineers are interested in the asymptotic stability of the zero solution of a delayed linear system. This can be seen from the references [9, 10], and [6, 11-15]. Partial differential equations are a powerful tool in modeling and analyzing dynamic systems in various scientific fields, including mathematical biology. In the context of biological systems, PDEs are often employed to describe the spatiotemporal evolution of different quantities, such as population densities, concentrations of biochemical species, or distributions of traits within a population. For more on this we refer to [16-18].

In many practical applications of delay differential equations, unbounded solutions are not physically meaningful or desirable. Boundedness of solutions ensures that the system's behavior remains within a certain range, which is often a physical requirement. For example, in control systems, unbounded solutions can lead to control signal saturation, which can be undesirable or even harmful. Thus, the boundedness of solutions in delay differential equations is a fundamental property that allows for the application of stability analysis techniques like Lyapunov's method. It ensures that the system's behavior remains well-behaved and within a certain range, which is not only mathematically important but also practically significant for understanding and controlling systems with time delays.

2. Stability

We begin by considering stability criteria for system (1.1). First, we introduce, a function V that we refer to as Lyapunov functional. For an extensive reading about various stability and the construction of Lyapunoiv functionals we refer to [3, 19, 20].

If $x(t)$ is any solution of system (1.1), then for a continuously differentiable function

$$V: R^+ \times R^n \to R^+,$$

we define the derivative $V^{'}$ of V by

$$V^{'}(t) := V^{'}(t,x) = \frac{\partial V(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t,x)}{\partial x_i} f_i(t,x_t),$$

where $f_i,\ i = 1,2,\ldots,3,n$ are the components of f in system (1.1). A continuous function $W: [0,\infty) \to [0,\infty)$ with $W(0) = 0$ is called a wedge if W strictly increasing. In this paper wedges are always denoted by W or W_i where i is a positive integer. For the next definition we denote C to be the set of continuous functions $\varphi: [-\alpha, 0] \to R^n,\ \alpha > 0$. If $t_0 = -\infty$, then the notation $t \geq t_0$ means $t > t_0$. It is to be understood that when

$$C(t) = \{\varphi: [t-\alpha, t] \to R^n\},$$

then $C(t)$ is C when $t = 0$. Also φ_t denotes $\varphi \in C(t)$ and $\| \phi_t \| = \max_{t-\alpha \leq s \leq t} |\phi(s)|$, where $|\cdot|$ is a convenient norm on R^n.

Definition 1 ([19]) Suppose $x(t) = 0$ is a solution of (1.1). (a) The zero solution of (1.1) is stable if for each $\varepsilon > 0$ and $t_1 \geq t_0$ there exists $\delta > 0$ such that $[\varphi \in C(t_1), \|\varphi\| < \delta, t \geq t_1]$ imply that $|x(t, t_1, \varphi)| < \varepsilon$. (b) The zero solution of (1.1) is uniformly stable if it is stable and if δ is independent of $t_1 \geq t_0$. (c) The zero solution of (1.1) is asymptotically stable if it is stable and if for each $t_1 \geq t_0$ there is an $\eta > 0$ such that $[\varphi \in C(t_1), \|\varphi\| < \eta]$ imply that $|x(t, t_1, \varphi)| \to 0$ as $t \to \infty$. Note that if this is true for every $\eta > 0$, then $x = 0$ is asymptotically stable in the large or globally asymptotically stable.

(d) The zero solution is uniformly asymptotically stable if it is (US) and if there exists a $\gamma > 0$ with the property that for each $\mu > 0$ there exists $T = T(\mu) > 0$ such that $[t_1 > t_0, \varphi \in C(t_1), \|\varphi\| < \gamma, t \geq t_1 + T]$ imply that $|x(t, t_1, \varphi)| < \mu$.

Another definition regarding boundedness of solutions is as follows.

Definition 2 We say that solutions of system (1.1) are bounded, if there exists a positive constant $D = D(|\phi|, t_0)$ such that any solution $x(t, t_0, \phi)$ of (1.1) satisfies

$$|x(t, t_0, \phi)| \leq D, \quad for\ all\ t \geq t_0,$$

where $D: R^+ \times R^+ \to R^+$ is a constant that depends on t_0 and ϕ. We say that solutions of system (1.1) are uniformly bounded if D is independent of t_0.

For more various types of stability and qualitative analysis of solutions of delay systems, we refer to [6, 21, 22] and the references therein. Our first result is given in the next theorem.

Theorem 1 Consider the neutral and totally nonlinear differential equation with bounded delay

$$x^{'}(t) = -ax^5(t) + bx^5(t-r), t \geq t_0 \geq 0 \tag{2.1}$$

where a, b and r are positive constants. If

$$b - a \leq 0,$$

then all solutions of (2.1) are bounded and its zero solution is uniformly stable. Moreover, $x^5(t) \in L^1[0, \infty)$.

Proof. Consider the Lyapunov type functional

$$V(x_t) = |x(t)| + b\int_{t-r}^{t} |x^5(s)|ds.$$

Assume $a \geq b$. Then along the solutions we have

$$\begin{aligned} V^{'}(x_t) &= \frac{x}{|x|}x^{'} - b|x^5(t-r)| + b|x^5(t)| = \frac{x}{|x|}(-ax^5(t) + bx^5(t-r)) - b|x^5(t-r)| + b|x^5(t)| \\ &\leq -a|x^5(t)| + b|x^5(t-r)| - b|x^5(t-r)| + b|x^5(t)| = [-a+b]|x^5| \leq 0. \end{aligned} \tag{2.2}$$

Let $t_1 \in R, \phi \in C(t_1)$ and $x(t) = x(t, t_1, \phi)$. Using the fact that $|x(t)| \leq V(x_t)$, and integrating (2.2) from $t_1 - r$ to t_1 yield

$$|x(t)| \leq V(x_t) \leq V(\phi_{t_1}) = |\phi(t_1)|^5 + b\int_{t_1-r}^{t_1} |\phi^5(s)|ds \leq (1+br)||\phi_{t_1}||^5. \tag{2.3}$$

Therefore, it follows from (2.3) that all solutions of (2.1) are bounded. As for the uniform stability, we let $\varepsilon > 0$ and take $\delta = (\frac{\varepsilon}{1+br})^{1/5}$. Then for $||\phi_{t_1}|| < \delta$, (2.3) yields that

$$|x(t)| \leq (1+br)[(\frac{\varepsilon}{1+br})^{1/5}]^5 = \varepsilon.$$

Thus, the zero solution is uniformly stable. We remark that the stability does not depend on the size of the delay r.

Next we integrate (2.2) for 0 to t and obtain $V(x_t) - V(\phi_0) \leq -\int_0^t |x(s)|^3 ds$, which implies that

$$\int_0^t |x(s)|^5 ds \leq V(\phi_0) - V(x_t) \leq V(\phi_0) < \infty,$$

for all $t \geq 0$. Hence $x^3(t) \in L^1[0, \infty)$.

It is worth noting that (2.3) does not necessary imply $|x(t, t_1, \phi)| \to 0$, as $t \to \infty$. Moreover, the results did not depend on the size of the delay r. Finally, we observe that the function $V = |x(t)|$ would not have produced $V^{'}(t) \leq 0$ along the solutions of (2.1). Also, Theorem 1, without any difficulty, can be extended to neutral equations with a time delay. For more on delay differential equations with oscillation properties we refer to [23-25].

3. General Theorems

In this section, we state a known theorem that provides sufficient conditions for various types of stability of the zero solution of (1.1). The proof of the next theorem can be found in [19].

Theorem 2 ([19]) Let $D > 0$ and there is a scalar functional $V(t, \psi_t)$ that is continuous in ψ and locally Lipschitz in ψ_t when $t \geq t_0$ and $\psi_t \in C(t)$ with $||\psi_t|| < D$. Suppose also that $V(t, 0) = 0$ and

$$W_1(|\psi(t)|) \leq V(t, \psi_t). \tag{3.1}$$

(a) If

$$V'(t, \psi_t) \leq 0 \ for \ t_0 \leq t < \infty \ and \ ||\psi_t|| \leq D, \tag{3.2}$$

then the zero solution of (1.1) is stable.

(b) If in addition to (a),

$$V(t, \psi_t) \leq W_2(||\psi_t||), \tag{3.3}$$

then the zero solution of (1.1) is uniformly stable.

(c) If there is an $M > 0$ with $|F(t, \psi_t)| \leq M \ for \ t_0 \leq t < \infty \ and \ ||\psi_t|| \leq D$, and if

$$V'(t, \psi_t) \leq -W_2(|\psi(t)|), \tag{3.4}$$

then the zero solution of (1.1) is asymptotically stable.

Part (c) of Theorem 2 requires the existence of a lower wedge on the Lyapunov function or functional, a requirement that is sometimes hard to fulfill. To get around it, we state and prove the following theorem, which is new for delay equations.

Theorem 3 Let $Q \subset R^n$ that is open with $0 \in Q$. Let $V: Q \to [0, \infty)$ be an autonomous function so that

$$V(0) = 0 \text{ and } V(x) > 0, \text{ if } x \neq 0.$$

Suppose that along the solutions of (1.1) we have

$$V'(x) \leq 0.$$

Let $x(s) = \phi(s), \ s \in [-h, 0], \ h > 0$. Assume each solution $x(t, t_1, \phi)$, which remains in Q exists on $[t_1, \infty)$. Then the zero solution of (1.1) is uniformly stable.

Proof. Let $\varepsilon > 0$ be given. The proof will be done if we can find $\delta > 0$ so that $[\phi \in C(t_1), |\phi| < \delta, t \geq t_1 \geq 0]$ imply that $|x(t, t_1, \phi)| < \varepsilon$. Let $\partial P(0, \varepsilon)$ be the boundary of the spherical neighborhood centered at the origin with radius ε. Since V is continuous and $\partial P(0, \varepsilon)$ is compact, then the minimum $m > 0$ of V exists in $\partial P(0, \varepsilon)$. Due to the fact that $V(0) = 0$, and V is continuous, there is a $\delta > 0$ such that $|\phi| < \delta$ implies $V(\phi) < m < \varepsilon$. Since solutions exist for all time in Q, we can find a $d > 0$ such that $x(t, t_1, \phi)$ exists for $t_1 \leq t < d$. Consequently, based on the hypothesis of the theorem, we have $V'(x(t, t_1, \phi)) \leq 0$. This gives

$$V'(x(t, t_1, \phi)) \leq V(\phi) < m$$

for as long as the solution is defined and in Q. However, the solution's path of $x(t, t_1, \phi)$ is connected and starts inside $P(0, \varepsilon)$, and $V(x) \geq m$ on $\partial P(0, \varepsilon)$. Thus, $x(t, t_1, \phi)$ could never reach $\partial P(0, \varepsilon)$. We conclude that $|x(t, t_1, \phi)| < \varepsilon$ for all future times, and hence $d = \infty$. This concludes the proof.

As an application of Theorem 3, we offer the following example.

Example 1 Let $h > 0$ and consider the neutral differential equation with finite delay

$$x'(t) = ax(t) + bx'(t-h) + cx(t-h),\ t \geq 0, \tag{3.5}$$

with $x(t_0) = \phi$. Suppose

$$2a + |c - ab| \leq 0, \quad |c - ab| - 2bc \leq 0.$$

Then the zero solution is uniformly stable.

Let

$$V(t, x_t) = (x(t) - bx(t-h))^2$$

and rewrite equation (3.5) as

$$\frac{d}{dt}[x(t) - bx(t-h)] = ax(t) + cx(t-h),\ t \geq 0.$$

Then along solutions of (3.5) we have

$$\begin{aligned} V'(t,x) &= 2(x(t) - bx(t-h))\frac{d}{dt}(x(t) - bx(t-h)) \\ &= 2ax^2(t) + 2(c - ab)x(t)x(t-h) - 2bcx^2(t-h). \end{aligned} \tag{3.6}$$

Using the inequality that for any two real numbers y and z, $yz \leq y^2/2 + z^2/2$, we have

$$2(c - ab)x(t)x(t-h) \leq |(c - ab)|(x^2(t) + x^2(t-h)).$$

Thus, expression (3.6) reduces to

$$V'(t,x) \leq (2a + |c - ab|)x^2(t) + (|c - ab| - 2bc)x^2(t-h) \leq 0.$$

Consequently, the result follows from Theorem 3.

It is natural to be interested in the asymptotic stability of the zero solution. The next theorem offers a criterion for uniform asymptotic stability. Dealing with delay systems with delay in the derivatives is difficult and tricky when finding a suitable Lyapunov functional. The next theorem provides sufficient conditions for the uniform asymptotic stability of the zero solution, and its proof can be found in [26]. The theorem assumes the following norm. For $\varphi \in C(t)$, we let

$$|||\varphi_t||| = [\sum_{i=1}^{n}\int_{t-\alpha}^{t}\varphi_i^2(s)]^{1/2}$$

where $\varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$.

Theorem 4 ([19]) Let $D > 0$ and there is a scalar functional $V(t, \varphi_t)$ that is continuous in ψ and locally Lipschitz in φ_t when $t \geq t_0$ and $\varphi_t \in C(t)$ with $||\varphi_t|| < D$. If V such that $V(t, 0) = 0$,

$$W_1(|\varphi(t)|) \leq V(t, \varphi_t) \leq W_2(|\varphi_t|) + W_3(|||\varphi_t|||),$$

and

$$\Delta V(t, \varphi_t) \leq -W_4(|\varphi(t)|),$$

then the zero solution of (1.1) is uniformly asymptotically stable .

As an application we offer the following example.

Example 2 Consider the system with multiple delays

$$x^{'}(t) = a(t)x(t) + \sum_{i=1}^{k} b_i(t)x(t-h_i), \tag{3.7}$$

where the delays h_i are positive integers for $i = 1,2,3,\dots,k$. Assume there are positive constants α and δ such that

$$2a(t) + \sum_{i=1}^{k} |b_i(t)| + k\delta \leq -\alpha \ and \ \delta \geq \sum_{i=1}^{k} |b_i(t)|.$$

Then the zero solution of (3.7) is (UAS). Consider the Lyapunov functional

$$V(t,x_t) = x^2(t) + \delta \sum_{i=1}^{k} \int_{t-h_i}^{t} x^2(s)ds. \tag{3.8}$$

Then along the solutions of (3.7) we have

$$\begin{aligned}
&V^{'}(t,x_t) = 2x(t)x^{'}(t) + \delta \textstyle\sum_{i=1}^{k}[x^2(t) - x^2(t-h_i)] \\
&= 2x(t)(ax(t) + \textstyle\sum_{i=1}^{k} \ b_i(t)x(t-h_i)) \\
&+\delta \textstyle\sum_{i=1}^{k}[x^2(t) - x^2(t-h_i)] \\
&\leq (2a(t) + \textstyle\sum_{i=1}^{k} b_i(t) + k\delta)x^2(t) \\
&+(\textstyle\sum_{i=1}^{k} |b_i(t)| - \delta)x^2(t-h_i) \\
&\leq -\alpha x^2(t).
\end{aligned}$$

To make sure the conditions of Theorem 4 are satisfied, we note that

$$\begin{aligned}
&x^2(t) \leq V(t,x_t) = x^2(t) + \delta \textstyle\sum_{i=1}^{k} \int_{t-h_i}^{t} x^2(s)ds \\
&= x^2(t) + \delta \textstyle\sum_{i=1}^{k} \int_{-h_i}^{0} x^2(u+t)du.
\end{aligned}$$

Hence, if we take

$$W_1(|\varphi(t)|) = W_2(|\varphi(t)|) = \varphi^2(t), \ W_4(|\varphi(t)|) = \alpha\varphi^2(t)$$

and

$$W_3(|\varphi_t|) = \delta \sum_{i=1}^{k} \int_{-h_i}^{0} \varphi^2(u+t)|du,$$

then we satisfy all the requirements of Theorem 4 and hence we have that the zero solution of (3.7) is (UAS). This completes the proof.

4. Boundedness and Exponential Stability

In this section we use non-negative Lyapunov type functions and establish sufficient conditions to obtain boundedness results on all solutions $x(t)$ of (1.1) and on the exponential stability when $x = 0$ is a solution. In the realm of control systems, for instance, exponential stability is a desirable trait because it guarantees that the system will reach a stable state efficiently and without unnecessary oscillations. This makes the system more predictable and easier to control, which is crucial in applications like robotics or electronic circuits.

From this point forward, if a function is written without its argument, then the argument is assumed to be t.

Definition 3 Assume $x=0$ is a solution of (1.1). That is $f(t,0)=0$. Then the zero solution of (1.1) is said to be α -exponentially asymptotically stable if there is a positive continuous function $\alpha(t)$ so that $\int_{t_0}^{t}\alpha(s)ds\to\infty \ \ as\ \ t\to\infty$ any solution $x(t,t_0,\phi)$ of (1.1) satisfies

$$|x(t,t_0,\phi)|\le C(|\phi|,t_0)(e^{-\int_{t_0}^{t}\alpha(s)ds})^{d},\quad for\ all\ t\in[t_0,\infty),$$

where d is a positive constant and $C\in R^{+}$. Moreover, the zero solution of (1.1) is said to be α - uniformly exponentially asymptotically stable if C is independent of t_0.

Theorem 5 Assume $D\subset R^{n}$ and there exists a type I Lyapunov function $V:D\to[0,\infty)$ such that for all $(t,x)\in[-h,\infty)\times D$:

$$W_1(||x||)\le V(t,x), \tag{4.1}$$

$$V'(t,x)\le-\alpha(t)V^{q}(t,x)+\beta(t), \tag{4.2}$$

$$V(t,x)-V^{q}(t,x)\le\gamma(t), \tag{4.3}$$

where $\alpha(t),\gamma(t)$ and $\beta(t)$ are non-negative and continuous functions; p, q are positive constants. Then all solutions of (1.1) that start in D satisfy

$$||x||\le W_1^{-1}\{V(t_0,\phi)e^{-\int_{t_0}^{t}\alpha(s)ds}+\int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{-\int_{u}^{t}\alpha(s)ds}\ du\}, \tag{4.4}$$

for all $t\ge t_0$.

We remark that in the case $q=1$, then the proof is a simple application of the variation of parameters formula. On the other hand, if $q\neq1$, using the variation of parameters formula will not be of much help. Now we prove the theorem.

Proof. For any initial time $t_0\ge0$, let $x(t,t_0,\phi)$ be any solution of (1.1) with $x_{t_0}=\phi$. Consider

$$\begin{aligned}
&\tfrac{d}{dt}(e^{\int_{t_0}^{t}\alpha(s)ds}V(t,x))\\
&=[V'(t,x)+\alpha(t)V(t,x)]e^{\int_{t_0}^{t}\alpha(s)ds}\\
&\le(-\alpha(t)V^{q}(t,x)+\beta(t)+\alpha(t)V(t,x))e^{\int_{t_0}^{t}\alpha(s)ds}\\
&\le(\alpha(t)(V(t,x)-V^{q}(t,x))+\beta(t))e^{\int_{t_0}^{t}\alpha(s)ds}\\
&\le(\alpha(t)\gamma(t)+\beta(t))e^{\int_{t_0}^{t}\alpha(s)ds}.
\end{aligned}$$

Integrating both sides from t_0 to t with $x_{t_0}=\phi$, we obtain, for $t\in[t_0,\infty)$,

$$V(t,x)e^{\int_{t_0}^{t}\alpha(s)ds}\le V(t_0,\phi)+\int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{\int_{t_0}^{u}\alpha(s)ds}\ du.$$

It follows that

$$V(t,x_t) \le V(t_0,\phi)e^{-\int_{t_0}^{t} \alpha(s)ds} + \int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{-\int_{u}^{t} \alpha(s)ds}\ du$$

for all $t \in [t_0,\infty)$. Inequality (4.1) implies that

$$||x(t)|| \le W^{-1}\{V(t_0,x_0)e^{-\int_{t_0}^{t} \alpha(s)ds} + \int_{t_0}^{t}(\gamma\alpha(u)+\beta(u))\ e^{-\int_{u}^{t} \alpha(s)ds}\ du\},$$

for all $t \ge t_0$. This concludes the proof.

Notice that that if $W_1(||x||) = ||x||^p$ for some positive p, then from inequality (4.1) we have that

$$||x|| \le \{V(t_0,\phi)e^{-\int_{t_0}^{t} \alpha(s)ds} + t_0t(\gamma(u)\alpha(u)+\beta(u))\ e^{-\int_{u}^{t} \alpha(s)ds}\ du\}^{1/p}. \tag{4.5}$$

Corollary 1 Assume the hypothesis of Theorem 5 hold.

(a) If the initial function ϕ is uniformly bounded and

$$\int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{-\int_{u}^{t} \alpha(s)ds}\ du \le M, \forall t \ge t_0 \ge 0, \tag{4.6}$$

for some positive constant M, then all solutions of (1.1) are uniformly bounded.

(b) If

$$f(t,0) = 0,$$

$$\int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{\int_{t_0}^{u} \alpha(s)ds}\ du \le M, \tag{4.7}$$

for some positive constant M, and

$$\int_{t_0}^{t}\alpha(s)ds \to \infty \quad as \quad t \to \infty \ for\ all \quad t \ge t_0, \tag{4.8}$$

then the zero solution of (1.1) is α -exponentially asymptotically stable with $d = 1/p$.

Proof. Let x be a solution to (1.1) that starts in D for all $t \ge t_0 \ge 0$. Hence the proof of *(a)* is an immediate consequence of inequality (4.5). For the proof of *(b)*, we consider the inequality from the proof of Theorem 5

$$V(t,x_t)e^{\int_{t_0}^{t} \alpha(s)ds} \le V(t_0,\phi) + \int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{\int_{t_0}^{u} \alpha(s)ds}\ du$$

for all $t \in [t_0,\infty)$. This yields

$$V(t,x_t) \le [V(t_0,\phi) + \int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{\int_{t_0}^{u} \alpha(s)ds}\ du]e^{-\int_{t_0}^{t} \alpha(s)ds}$$

for all $t \in [t_0,\infty)$. Using $W_1(||x||) = ||x||^p$, we have

$$||x(t)|| \le [V(t_0,\phi) + \int_{t_0}^{t}(\gamma(u)\alpha(u)+\beta(u))\ e^{\int_{t_0}^{u} \alpha(s)ds}\ du]^{1/p}e^{-\frac{1}{p}\int_{t_0}^{t} \alpha(s)ds}.$$

This concludes the proof.

Remark 1 If $f(t,0) \neq 0$ and (4.7) and (4.8) hold, then all solutions of (1.1) decay exponentially to zero.

Next, we furnish examples in the form of theorems since all the results are fundamental and new to such neutral delay equations. In the next four theorems we assume the existence of an initial function ϕ that is continuous on $[-h,0]$.

Theorem 6 Let $h > 0$ and consider the neutral differential equation with finite delay

$$x'(t) = a(t)x(t) + b(t)x'(t-h) + g(t),\ t \geq 0, \tag{4.9}$$

with $x(t_0) = \phi$. Suppose $a(t), g(t)$ are continuous for all $t \geq 0$ and for all $t \geq 0$, $b(t)$ is continuously differentiable. Assume for all $t \geq 0$, that

$$b'(t) + (1 + a(t))b(t) = 0, \tag{4.10}$$

$$2a(t) + 1 + |g(t)| \leq 0, \tag{4.11}$$

$$2b(t)b'(t) + b^2(t) + |g(t)| \leq 0, \tag{4.12}$$

and there exists a positive constant L such that

$$|g(t)| + b^2(t)|g(t)| \leq L,\ t \geq 0.$$

Then $|x(t) - b(t)x(t-h)|$ is bounded and decays exponentially to zero.

Proof. Let

$$V(t,x_t) = (x(t) - b(t)x(t-h))^2$$

and rewrite equation (4.9) as

$$\frac{d}{dt}[x(t) - b(t)x(t-h)] = a(t)x(t) - b'(t)x(t-h) + g(t),\ t \geq 0.$$

Then along solutions of (4.9) we have

$$\begin{aligned} V'(t,x) &= 2(x(t) - b(t)x(t-h))\frac{d}{dt}(x(t) - b(t)x(t-h)) \\ &= 2a(t)x^2(t) - 2(b'(t) + a(t)b(t))x(t)x(t-h) \\ &+2b(t)b'(t)x^2(t-h) + 2x(t)g(t) - 2b(t)x(t-h)g(t). \end{aligned} \tag{4.13}$$

Using the inequality $yz \leq y^2/2 + z^2/2$ for any two real numbers y and z we have

$$2x(t)g(t) \leq 2|x(t)|\ |g(t)| = 2|x(t)|\ |g(t)|^{1/2}|g(t)|^{1/2} \leq x^2(t)|g(t)| + |g(t)|.$$

A similar work on $-2b(t)x(t-h)g(t)$ leads to

$$-2b(t)x(t-h)g(t) \leq x^2(t-h)|g(t)| + b^2(t)|g(t)|.$$

Thus, expression (4.13) reduces to

$$\begin{aligned} V'(t,x) &\leq (2a(t) + |g(t)|)x^2(t) - 2(b'(t) + a(t)b(t))x(t)x(t-h) \\ &+(2b(t)b'(t) + |g(t)|)x^2(t-h) + |g(t)| + b^2(t)|g(t)|. \end{aligned}$$

We observe that (4.3) is immediate since since $q = 1$. Left to show condition (4.2) hold. Using conditions (4.10)-(4.12) we have

$$V(t,x_t)+V^{'}(t,x_t)=(2a(t)+1+|g(t)|)x^2(t)+(2b(t)b^{'}(t)+b^2(t)$$
$$+|g(t)|)x^2(t-h)+|g(t)|+b^2(t)|g(t)|$$
$$-2(b^{'}(t)+a(t)b(t)+b(t))x(t)x(t-h)$$
$$\leq |g(t)|+b^2(t)|g(t)|:=\gamma.$$

Inequality (4.1) of Theorem 5 yields,

$$(x(t)-b(t)x(t-h))^2 \leq V(t_0,\phi)e^{-(t-t_0)}+\gamma$$
$$\leq (\phi(t_0)-b(t_0)\phi(t_0-h))^2e^{-(t-t_0)}+\gamma.$$

Taking the square root we obtain

$$|x(t)-b(t)x(t-h)| \leq [(\phi(t_0)-b(t_0)\phi(t_0-h))^2e^{-(t-t_0)}+\gamma]^{1/2},\ for\ t\geq t_0.$$

Upon closely examining (4.12), we find that substituting $u=b^2(t)$ yields the linear differential equation $\frac{du}{dt}+u=-|g(t)|$, which yields $b(t)=(ce^{-t}-e^{-t}\int_0^t|g(s)|e^sds)^{1/2}$,where c is an appropriate constant that ensures that $b(t)$ is well defined for all $t\geq 0$. Taking $c=1, a=-1$, and $g(t)=e^{-2t}$, for instance, makes it simple to check the conditions (4.10), (4.11), and (4.12).

We have the following corollary.

Corollary 2 If $g(t)=0$ for all $t\geq 0$, then $\gamma=0$, and the conditions of Theorem 6 take the form

$$b^{'}(t)+(1+a(t))b(t)=0,$$

$$2a(t)+1\leq 0,$$

$$2b(t)b^{'}(t)+b^2(t)\leq 0.$$

The we have

$$|x(t)-b(t)x(t-h)| \leq (1+|b(t_0)|)||\phi||e^{-\frac{1}{2}(t-t_0)},\ for\ t\geq t_0. \tag{4.14}$$

Remark 2 Since $b(t)$ is bounded, and if the initial function is uniformly bounded, then from Theorem 6 and Corollary 2 we deduce that the boundedness and the exponential decay are uniform.

Remark 3 The condition (4.11) is somewhat onerous. For $k\geq 2.$, it necessitates that the function $g(t)$ decay exponentially on the order of $g(t)=e^{-kt}$. Assuming that $g(t)$ is bounded above and does not have to decay to zero as $t\to\infty$, one anticipates that all solutions of (4.9) are bounded, with additional assumptions on the coefficients $a(t), b(t)$, and $b'(t)$. In the next corollary, we review $V^{'}(t,x_t)$ in order to alleviate this restriction.

Corollary 3 Consider equation (4.9) and assume the hypothesis of Theorem 6 holds along with condition (4.10). Suppose

$$a(t)+1\leq 0, \tag{4.15}$$

$$2b(t)b^{'}(t)+b^2(t)\leq 0, \tag{4.16}$$

and there exists a positive constant L such that

$$|g(t)|\leq L,\ t\geq 0.$$

Then all solutions of (4.9) are uniformly bounded.

Proof. Let $V(t,x_t)$ be defined as in Theorem 6. We recalculate the expression $V^{'}(t,x_t)$. We revisit the following two terms.

$$2x(t)g(t) \leq 2|x(t)|\ |g(t)| \leq x^2(t) + g^2(t).$$

A similar work on $-2b(t)x(t-h)g(t)$ leads to

$$-2b(t)x(t-h)g(t) \leq b^2(t)x^2(t-h) + g^2(t).$$

Then, expression (4.13) reduces to

$$V^{'}(t,x) \leq (2a(t)+1)x^2(t) - 2(b^{'}(t) + a(t)b(t))x(t)x(t-h)$$
$$+(2b(t)b^{'}(t) + b^2(t))x^2(t-h) + 2g^2(t).$$

Thus

$$V(t,x_t) + V^{'}(t,x_t) = 2(a(t)+1)x^2(t) + (2b(t)b^{'}(t) + 2b^2(t))x^2(t-h) + 2g^2(t)$$
$$-2(b^{'}(t) + (1+a(t))b(t))x(t)x(t-h)$$
$$\leq 2g^2(t) := \gamma.$$

The results are obtained by invoking Theorem 5.

Again, we note that condition (4.16) yields that the function $b(t)$ is of the form $b(t) = ke^{-t}$, for some suitable positive constant k. Also, in Corollary 3 the function $g(t)$ has to be only bounded.

The results of Theorem 6 and Corollary 3 have profound usage in the study of control theory. For example, in [27] the author considered the control system

$$\frac{dx(t)}{dt} - c\frac{dx(t-\tau(t))}{dt} = ax(t) + bx(t-\tau(t)), \quad x(t_0) = x_0. \tag{4.17}$$

By letting

$$y(t) = x(t) - cx(t-\tau(t))$$

system (4.17) is transferred to

$$\frac{dy(t)}{dt} = ay(t) + (ac+b)x(t-\tau(t)), \quad y(t_0) = x_0 - c\phi(-\tau). \tag{4.18}$$

In the papers [6, 11–15] it was argued that if the zero solution of $\frac{dy(t)}{dt} = ay(t)$ is asymptotically stable, then the zero solution of (4.18) is asymptotically stable provided that $|ac+b| < 1$.

In the next theorem, we consider a delay differential equation with forcing function f and display an unusual Lyapunov functional to prove boundedness on all solutions and exponential asymptotic stability in the case $f(t) = 0$, for all $t \geq 0$. The type of Lyapunov functional that we display here does not require constraints on the delay, instead the emphasis will be on the size of the coefficients.

Theorem 7 For positive h, we consider the delay equation

$$x^{'}(t) = a(t)x(t) + b(t)x(t-h) + f(t),\ t \geq 0, \tag{4.19}$$

with $x(t_0) = \phi$. Suppose $a(t), b(t)$ and $f(t)$ are continuous for all $t \geq 0$.

Assume

$$2a(t) + 3 + |b(t)| \leq 0, \tag{4.20}$$

$$|b(t)| - e^{-h} \leq 0, \tag{4.21}$$

and there exists a positive constant L such that

$$|f(t)| \leq M, \ t \geq 0.$$

Then all solutions of (4.19) are uniformly bounded. Moreover, if $f(t) = 0$, for all $t \geq 0$, then the zero solution is exponentially asymptotically stable.

Proof. For $t \geq 0$, let

$$V(t, x_t) = x^2(t) + \int_{t-h}^{t} e^{-(t-s)} x^2(s) \ ds.$$

Then along solutions of (4.19) we have

$$V^{'}(t, x) = 2a(t)x^2(t) + 2b(t)x(t)x(t-h) + 2x(t)f(t) + x^2(t)$$
$$-e^{-h}x^2(t-h) - \int_{t-h}^{t} e^{-(t-s)} x^2(s) \ ds$$
$$\leq (2a(t) + 2 + |b(t)|)x^2(t) + (|b(t)| - e^{-h})x^2(t-h)$$
$$- \int_{t-h}^{t} e^{-(t-s)} x^2(s) \ ds + f^2(t).$$

Using conditions (4.20) and (4.21) we have

$$V(t, x_t) + V^{'}(t, x_t) \leq (2a(t) + 3 + |b(t)|)x^2(t) + (|b(t)| - e^{-h})x^2(t-h)$$
$$- \int_{t-h}^{t} e^{-(t-s)} x^2(s) \ ds + \int_{t-h}^{t} e^{-(t-s)} x^2(s) \ ds + f^2(t)$$
$$\leq f^2(t) := \gamma.$$

It follows from inequality (4.4) that

$$|x| \leq [V(t_0, \phi)e^{-(t-t_0)} + \gamma]^{1/2}$$
$$\leq [(1 + \int_{t_0-h}^{t_0} e^{-(t_0-s)} \ ds)||\phi||^2 \ e^{-(t-t_0)} + \gamma]^{1/2}$$
$$\leq [(2 - e^{-h})||\phi||^2 \ e^{-(t-t_0)} + \gamma]^{1/2}.$$

This gives boundedness. Now if $f(t) = 0$, for all $t \geq 0$, then $\gamma = 0$ and hence the above inequality gives the exponential stability.

As we know, conditions that ensure stability, depend on the Lyapunov functional, as the next corollary shows.

Corollary 4 Assume the hypothesis of Theorem 7 hold, where condition (4.20) is replaced with

$$a(t) + 2 \leq 0.$$

Then all solutions of (4.19) are uniformly bounded. Moreover, if $f(t) = 0$, for all $t \geq 0$, then the zero solution is exponentially asymptotically stable.

Proof. The proof is achieved by taking

$$V(t, x_t) = |x(t)| + \int_{t-h}^{t} e^{-(t-s)} |x(s)| \ ds.$$

The next theorem deals with distributed delays.

Theorem 8 Let $h > 0$ and consider the functional delay differential equation

$$x'(t) = a(t)x(t) + b(t)x(t-h) + e^{-2t}\int_{t-h}^{t} C(s)x(s)ds + g(t),\ t \geq 0, \tag{4.22}$$

with $x(t_0) = \phi$ and all the functions are continuous for all $t \geq 0$.

Assume

$$a(t) + 1 + \int_{t-h}^{\infty} |C(u+t)|\ du \leq 0, \tag{4.23}$$

$$|b(t)| - e^{-h}\int_{t-h}^{\infty} |C(u+t-h)|\ du \leq 0, \tag{4.24}$$

$$e^{-2t}|C(s)| - e^{-(t-s)}|C(t+s-h)| \leq 0,\ for\ all\ t-h \leq s \leq t,\ t \geq 0, \tag{4.25}$$

and there exists a positive constant L such that

$$|g(t)| \leq L,\ t \geq 0.$$

Then all solutions of (4.22) are uniformly bounded provided that

$$\int_{t-h}^{t}\int_{t-h}^{\infty} e^{-(t-s)}\ |C(u+s)|\ du\ ds < \infty.$$

Moreover, if $g(t) = 0$ for all $t \geq 0$, then the zero solution of (4.22) is exponentially asymptotically stable.

Proof. For $t \geq 0$, let

$$V(t, x_t) = |x(t)| + \int_{t-h}^{t}\int_{t-h}^{\infty} e^{-(t-s)}\ |C(u+s)|\ du\ |x(s)|\ ds.$$

Then along solutions of (4.22) we have

$$\begin{aligned}
V'(t,x) &\leq a(t)|x(t)| + |b(t)||x(t-h)| + e^{-2t}\textstyle\int_{t-h}^{t}|C(s)||x(s)|\ ds + |g(t)| \\
&+ \textstyle\int_{t-h}^{\infty}|C(u+t)|\ du\ |x(t)| - \int_{t-h}^{\infty} e^{-h}\ |C(u+t-h)|\ du\ |x(t-h)| \\
&- \textstyle\int_{t-h}^{t} e^{-(t-s)}\ |C(t+s-h)||x(s)|\ ds \\
&- \textstyle\int_{t-h}^{t}\int_{t-h}^{\infty} e^{-(t-s)}\ |C(u+s)|\ du\ |x(s)|\ ds.
\end{aligned}$$

Using conditions (4.23), (4.24) and (4.25), we have

$$\begin{aligned}
V(t,x_t) + V'(t,x_t) &\leq (a(t) + 1 + \textstyle\int_{t-h}^{\infty}|C(u+t)|\ du)|x(t)| \\
&+ (|b(t)| - e^{-h}\textstyle\int_{t-h}^{\infty}|C(u+t-h)|\ du)|x(t-h)| \\
&+ \textstyle\int_{t-h}^{t}[e^{-2t}|C(s)| - e^{-(t-s)}|C(t+s-h)|]|x(s)|\ ds + |g(t)| \\
&\leq L.
\end{aligned}$$

By invoking Theorem 5 we arrive at boundedness result. Now, we set $g(t) = 0$ for all $t \geq 0$, which implies that $L = 0$ and hence by Theorem 8 the zero solution of (4.22) is exponentially asymptotically stable.

Remark 4 All the requirements of Theorem 8 are satisfied if we let $C(t) = e^{-t}$, given that $g(t)$ is bounded and $|b(t)| \leq e^{h-2t}$ for all $t \geq 0$.

The author Wang in [28], considered (4.19) and provided sufficient conditions concerning uniform asymptotic stability of the zero solution. The motivation behind his work was to arrive at the asymptotic stability without requiring that $a(t) \leq 0$ for all $t \geq 0$. This was achieved by using the negativeness of $b(t)$ to offset the positiveness of $a(t)$. There was a price to pay, which is the limitation on the size of the delay h in order to satisfy the sufficient condition

$$-\frac{1}{2h} \leq a(t) + b(t+h) \leq -h\ b^2(t+h).$$

For example, if we set $a(t) = -2$ and $b(t) = e^{-h}$, then all the conditions of Theorem 7 are satisfied regardless of the size of the delay h. On the other hand, if we set $h = 2$, then the above inequality will not hold. For fairness, Wang's inequality does not ask for requires $a(t) \leq 0$ for all $t \geq 0$.

So far, we have been applying Theorem 5 with $\alpha(t)$ equal to a constant and $q = 1$. In the next theorem, we will display a Lyapunov functional for (4.19), in which the $\alpha(t)$ of Theorem 5 depends on time. This permits us to consider functions $g(t)$ that may not be bounded and yet still obtain boundedness of solutions and exponential stability of the zero solution.

Theorem 9 Consider (4.19) and define a continuous function $\eta(t) \geq 0$ such that for some $\tau > 0$

$$\eta(t) = \frac{e^{\int_0^t c(s)ds}}{1 + 2h\int_t^{t+\tau} e^{\int_0^u c(s)ds} du}, \tag{4.26}$$

where $c(t) := 2a(t) + |g(t)|$. Let

$$|b(t) \leq h\eta(t). \tag{4.27}$$

Then every solution of (4.19) with $x(t_0) = \phi$ satisfies the inequality

$$||x(t)|| \leq [V(t_0, \phi) + \int_{t_0}^t |g(u)|\ e^{\int_{t_0}^u \alpha(s)ds}\ du]^{1/2} e^{-\frac{1}{2}\int_{t_0}^t \alpha(s)ds}, \tag{4.28}$$

where

$$\alpha(t) = c(t) + 2h\eta(t) \ \ and \ \ V(t_0, \phi) = \phi^2(0) + h\eta(t_0)\int_{-h}^0 \phi^2(s)ds.$$

Proof. For $t \geq 0$, let

$$V(t, x_t) = x^2(t) + h\eta(t)\int_{t-h}^t x^2(s)ds$$

Then along solutions of (4.19) we have

$$\begin{aligned} V^{'}(t, x_t) &= 2a(t)x^2(t) + 2b(t)x(t)x(t-h) + 2x(t)g(t) + h\eta(t)x^2(t) \\ &\quad -h\eta(t)x^2(t-h) + h\eta^{'}(t)\int_{t-h}^t x^2(s)\ ds. \end{aligned} \tag{4.29}$$

As before,

$$2x(t)g(t) \leq 2|x(t)|\ |g(t)| = 2|x(t)|\ |g(t)|^{1/2}|g(t)|^{1/2} \leq x^2(t)|g(t)| + |g(t)|,$$

and

$$2b(t)x(t)x(t-h) \leq x^2(t-h)|b(t)| + x^2(t)|b(t)|.$$

Also, a simple differentiation in (4.26) yields to

$$\eta'(t) = c(t)\eta(t) + 2h\eta^2(t) - 2h\eta^2(t)e^{\int_t^{t+\beta} c(s)ds}$$
$$\leq c(t)\eta(t) + 2h\eta^2(t).$$

As a consequence, (4.28) reduces to

$$V'(t, x_t) \leq (c(t) + |g(t)|)V(t, x_t) + |g(t)|$$
$$= \alpha(t)V(t, x_t) + |g(t)|. \quad (4.30)$$

Applying the variation of parameters formula on (4.30) gives the desired inequality (4.28).

If there is a positive constant M so that

$$\int_{t_0}^{t} |g(u)|\, e^{\int_{t_0}^{u} \alpha(s)ds}\, du \leq M, \text{ forall} t \geq t_0,$$

then it follows from (4.28) that all solutions of (4.19) are bounded and uniformly bounded if the initial function is uniformly bounded. Additionally, if $g = 0$, then the zero solution (4.19) is exponentially stable.

5. Conclusion

The key findings of this research include the establishment of general theorems concerning the boundedness and various forms of stability of the zero solution, with a particular emphasis on achieving exponential stability. The attainment of exponential stability was shown to demand meticulous attention in the construction of a suitable Lyapunov functional. The theorems presented in this paper, specifically Theorems 5 and 9, offer constructive approaches for building such Lyapunov functions tailored to nonlinear delay systems. These findings contribute valuable insights into the intricate dynamics of these systems and provide a foundation for further exploration in the field of stability analysis. In conclusion, this paper not only extends the theoretical understanding of boundedness and stability but also offers practical tools through the provided theorems. The emphasis on constructing Lyapunov functionals for nonlinear delay systems adds depth to the existing knowledge, paving the way for advancements in control theory and related areas. The insights gained from this research hold significance in addressing real-world problems where stability considerations play a crucial role.

Conflict of Interest

The authors have no competing interests to declare that are relevant to the content of this article.

Funding

No funding was received for conducting this study.

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