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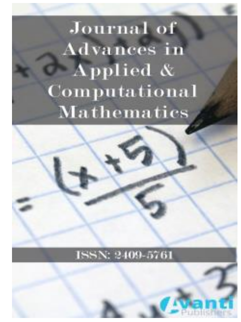

Solving System of Mixed Ordered Variational Inequalities Involving XOR and XNOR Operations in Ordered Product Banach Space

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ABSTRACT

In this article, we study a generalized system of mixed ordered variational inequalities problems with various operations in a real ordered product Banach space and discuss the existence of the solution of our considered problem. Further, we discuss the convergence analysis of the proposed iterative algorithm using XNOR and XOR operations techniques. Most of the variational inequalities solved by the projection operator technique but we solved our considered problem without the projection technique. The results of this paper are more general and new than others in this direction. Finally, we give a numerical example to illustrate and show the convergence of the proposed algorithm in support of our main result has been formulated by using MATLAB programming.

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1. Introduction

Variational inequalities have been generalized and extended in different directions using novel and innovative techniques and are applicable to solve many problems related to optimization and control, transportation equilibrium and economic, engineering, and basic sciences. For the recent state of the art [1-17]. We would like to point out that the projection method technique used to find the solution of variational inequalities which are quite general and flexible method.

In 1972, Amann [1] established for computing the solutions of nonlinear equations and fixed point theory with nonlinear mapping and applications have been studied with nonlinear increasing operators in real ordered Hilbert space or Banach spaces investigated by Du [18] which is applicable in nonlinear analysis and developed the methods to solve original mathematical problems. Future, many authors discussed and studied the idea of ordered nonlinear variational inequalities (inclusions) in different settings which is available in the literature [19-37].

In 2008, Li and his coauthors have investigated and analyzed the ordered variational inequality problem to obtain $u \in B$ such that $T(h(u)) \geq 0$ and after that introduced and studied a general nonlinear ordered variational inequalities problem to obtain $u \in B$ such that $A(u) \oplus B(u, h(u)) \geq 0$ (h, A and $B(.,.)$are nonlinear mappings), and discussed the existence and convergence results in real ordered Hilbert or Banach spaces with the help of restricted-accretive mapping techniques [38-39]. Very recently, many authors have been considered and studied ordered equations (inclusions) problem which solved by using the several kinds of single-valued (multiple-valued) mappings to find the solutions of ordered variational inequality (inclusions) with $\oplus$ operations in different settings [2,3, 38-43].

Inspired and motivated by ongoing research in this direction, the main aim of this paper is as follows. In section 2, contains the basic results needed in this paper. In Section 3, we consider a SMOVIP with various operations and prove the existence of a solution to the considered problem. In Section 4, we propose the iterative algorithms which are more general than the previous iterative algorithms investigated by many authors in the literature and analyze the convergence criteria of the proposed algorithm. Finally, we demonstrate a numerical example that satisfies all the conditions and show the convergence of the proposed algorithm of our main result.

2. Preliminaries

Let B be a real ordered Banach space with its norm $P.P$. Assume K is a normal cone with normal constant δ_K, and $\leq$ is a partial ordering defined by for arbitrary $u, v \in B$, $u \leq v$ if and only if $v - u \in K$. For any elements $u, v \in B$, $lub\{u, v\}$ and $glb\{u, v\}$ are denoted by least upper bound and greatest lower bound of the set $\{u, v\}$, respectively. Suppose $glb\{u, v\}$ and $lub\{u, v\}$ exist, some binary operations are defined as follows:

1. $u \vee v = sup\{u, v\}$;
2. $u \wedge v = inf\{u, v\}$;
3. $u \oplus v = (u - v) \vee (v - u)$;
4. $uev = (u - v) \wedge (v - u)$.

The operations $\vee$,$\wedge$, $\oplus$ and e are called AND, OR, XNOR and XOR operations, respectively.

Definition 2.1 (15,18) Let $K(\neq \emptyset) \subseteq B$. Then

1. K is called normal cone if and only if there exists a constant $\delta_K > 0$ such that for $0 \leq u \leq v$, we have $||u|| \leq \delta_K ||v||$, for any $u, v \in B$;
2. For any $u, v \in B$ if either $v \leq u$ or $u \leq v$ hold, then u and v are said to be comparable to each other (denoted by $u \propto v$).

Definition 2.2 (18,39) Let $P: B \to B$ be a single-valued mapping. Then

1. P is said to be a strongly comparison mapping, if P is a comparison mapping and $u \propto v$ if and only if $P(u) \propto P(v)$, for all $u, v \in B$;

2. a comparison mapping P is said to be a μ_P-ordered compression mapping, if there exists $0 < \mu_P < 1$ such that

$$P(u) \oplus P(v) \leq \mu_P(u \oplus v), for\ all\ u, v \in B.$$

Definition 2.3 A single-valued mapping $G: B \times B \times B \to B$ is called (κ, ν, τ) -ordered Lipschitz continuous, if $a \propto u$, $b \propto v$ and $c \propto w$, then $G(a, b, c) \propto G(u, v, w)$ and there exist constants $\kappa, \nu, \tau > 0$ such that

$$G(a, b, c) \oplus G(u, v, w) \leq \kappa(a \oplus u) + \nu(b \oplus v) + \tau(c \oplus w), for\ all\ a, b, c, u, v, w \in B.$$

Definition 2.4 (39) A compression mapping $J: B \to B$ is said to be restricted accretive mapping if there exist two constants $\xi_1, \xi_2 \in (0,1]$ such that

$$(J(u) + I(u)) \oplus (J(v) + I(v)) \leq \xi_1(J(u) \oplus J(v)) + \xi_2(u \oplus v), for\ any\ u, v \in B$$

holds, where I is an identity mapping on B.

Definition 2.5 (38) Let $J: B \to B$ be a single-valued mapping. A single-valued mapping $A: B \to B$ is said to be J -restricted accretive mapping if A, J and $A \wedge J$ all are comparisons with each other, and there exist two constants $\xi_1, \xi_2 \in (0,1]$ such that for any $u, v \in B$

$$(A(u) \wedge J(u) + I(u)) \oplus (A(v) \wedge J(v) + I(v)) \leq \xi_1((A(u) \wedge J(u)) \oplus (A(v) \wedge J(v))) + \xi_2(u \oplus v),$$

holds, where I is an identity mapping on B.

Definition 2.6 (39) Let $B \times B \times B$ be an real ordered product Banach space with the norm $P.P$ and an partial ordered relation $\leq$, and the following conditions are satisfied: for any $(w_1, w_2, w_3), (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3) \in B \times B \times B$

1. $(w_1, w_2, w_3) \leq (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3)$ if and only if $w_1 \leq \widehat{w}_1, w_2 \leq \widehat{w}_2$ and $w_3 \leq \widehat{w}_3$ in B;
2. $(w_1, w_2, w_3) \propto (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3)$ if and only if $w_1 \propto \widehat{w}_1, w_2 \propto \widehat{w}_2, w_3 \propto \widehat{w}_3$,
3. $(w_1, w_2, w_3) \wedge (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3) = (w_1 \wedge \widehat{w}_1, w_2 \wedge \widehat{w}_2, w_3 \wedge \widehat{w}_3)$, $(w_1, w_2, w_3) \vee (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3) = (w_1 \vee \widehat{w}_1, w_2 \vee \widehat{w}_2, w_3 \vee \widehat{w}_3)$, $(w_1, w_2, w_3) \oplus (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3) = (w_1 \oplus \widehat{w}_1, w_2 \oplus \widehat{w}_2, w_3 \oplus \widehat{w}_3)$.

Definition 2.7. For arbitrary sequences $\{u_n\}, \{v_n\}$, and $\{w_n\}$ in B, and the sequence $\{(u_n, v_n, w_n)\}$ in $B \times B \times B$,

$$u_n \to u^*, v_n \to v^*\ and\ w_n \to w^*\ if\ and\ only\ if\ (u_n, v_n, w_n) \to (u^*, v^*, w^*), as\ n \to \infty.$$

Definition 2.8 (38) A vector-valued mapping $\vec{F} = (F_1, F_2, F_3)\ (or\ (F_1, F_2, F_3)^T): B \times B \times B \to B \times B \times B$ in $B \times B \times B$, if there exists a point $(u^*, v^*, w^*) \in B \times B \times B$ such that

$$\vec{F}(u^*, v^*, w^*) = (F_1, F_2, F_3)(u^*, v^*, w^*) = (u^*, v^*, w^*),$$

holds, then (u^*, v^*, w^*) is called a fixed point of vector-valued mapping $\vec{F}$ in ordered product Banach space.

Definition 2.9 (38) A vector-valued mapping $\vec{F} = (F_1, F_2, F_3)\ (or\ (F_1, F_2, F_3)^T): B \times B \times B \to B \times B \times B$ in $B \times B \times B$, if for any $(u_j, v_j, w_j) \in B \times B \times B\ (j = 1,2)$, $(u_1, v_1, w_1) \propto (u_2, v_2, w_2)$ and there exists a constant $\zeta \in (0,1)$ such that

$$P(F_1, F_2, F_3)(u_1, v_1, w_1) \oplus (F_1, F_2, F_3)(u_2, v_2, w_2)P \leq \zeta P(u_1, v_1, w_1) \oplus (u_2, v_2, w_2)P,$$

then $\vec{F} = (F_1, F_2, F_3)$ has a fixed point in ordered product Banach space.

Lemma 2.1 (4,15,18,38) Let $\oplus$ and e be the XOR and XNOR operations, respectively. Then the following relations hold:

1. $wew = 0$, $wev = vew = -(w \oplus v) = -(v \oplus w)$;
2. $(\lambda w) \oplus (\lambda v) = |\lambda|(w \oplus v)$;
3. if $w \propto v$, then $w \oplus v = 0$ if and only if $w = v$;
4. $(u+v)e(w+z) \geq (uew) + (vez)$;
5. if w, u and v are comparative to each other, then $(w \oplus u) \leq w \oplus v + v \oplus u$;
6. if $w \propto v$, then $((w \oplus 0) \oplus (v \oplus 0)) \leq (w \oplus v) \oplus 0 = w \oplus v$;
7. $(lw) \oplus (mw) = |l - m|w = (l \oplus m)w$, for all $u, v, w, z \in B$ and $l, m, \lambda \in R$.

We construct the following example in support of restricted accretive mapping and J -restricted accretive mapping.

Example 2.1 Let $B = R$ and the single-valued mapping $J: B \to B$ is defined by

$$J(w) = \frac{1}{2} - \frac{w}{3}, \forall\, w \in B.$$

We can obtain the following expressions: $J(w) + I(w) = \frac{1}{2} + \frac{2w}{3}$ and

$$J(w) \oplus J(v) = (\tfrac{1}{2} - \tfrac{w}{3}) \oplus (\tfrac{1}{2} - \tfrac{v}{3})$$

$$= ((\tfrac{1}{2} - \tfrac{w}{3}) - (\tfrac{1}{2} - \tfrac{v}{3})) \vee ((\tfrac{1}{2} - \tfrac{v}{3}) - (\tfrac{1}{2} - \tfrac{w}{3}))$$

$$= (\tfrac{w}{3} - \tfrac{v}{3}) \vee (\tfrac{v}{3} - \tfrac{w}{3})$$

$$= (\tfrac{w}{3} \oplus \tfrac{v}{3})$$

$$= \tfrac{1}{3}(w \oplus v),$$

i.e.,

$$J(w) \oplus J(v) = \frac{1}{3}(w \oplus v).$$

Now,

$$(J(w) + I(w)) \oplus (J(v) + I(v)) = (\tfrac{1}{2} + \tfrac{2w}{3}) \oplus (\tfrac{1}{2} + \tfrac{2v}{3})$$

$$\leq (\tfrac{2w}{3} \oplus \tfrac{2v}{3})$$

$$= \tfrac{2}{3}(w \oplus v)$$

$$\leq (w \oplus v)$$

$$= \tfrac{3}{4}(J(w) \oplus J(v)) + \tfrac{3}{4}(w \oplus v),$$

i.e.,

$$(J(w) + I(w)) \oplus (J(v) + I(v)) \leq \frac{3}{4}(J(w) \oplus J(v)) + \frac{3}{4}(w \oplus v).$$

Therefore, J is restricted accretive mapping with constants $\xi_1 = \frac{3}{4}$ and $\xi_2 = \frac{3}{4}$, respectively.

Suppose the mapping $A: B \to B$ is defined by

$$A(w) = \frac{1}{4} - \frac{w}{6}, \forall\, w \in B.$$

Now,

$$A(w) \wedge J(w) = (\frac{1}{2} - \frac{w}{3}) \wedge (\frac{1}{4} - \frac{w}{6}) = inf\{\frac{1}{2} - \frac{w}{3}, \frac{1}{4} - \frac{w}{6}\} = \frac{1}{4} - \frac{w}{6}.$$

and

$$(A(w) \wedge J(w)) \oplus (A(v) \wedge J(v)) = \frac{5}{6}(w \oplus v).$$

$(A(w) \wedge J(w) + I(w)) \oplus (A(v) \wedge J(v) + I(v)) = (\frac{1}{4} + \frac{5w}{6}) \oplus (\frac{1}{4} + \frac{5v}{6})$

$= ((\frac{1}{4} + \frac{5w}{6}) - (\frac{1}{4} + \frac{5v}{6}))$

$\vee ((\frac{1}{4} + \frac{5w}{6}) - (\frac{1}{4} + \frac{5v}{6}))$

$= (\frac{5w}{6} - \frac{5v}{6}) \vee (\frac{5v}{6} - \frac{5w}{6})$

$= \frac{5}{6}(w \oplus v)$

$\leq (w \oplus v)$

$= \frac{2}{5}((A(w) \wedge J(w)) \oplus (A(v) \wedge J(v)))$

$+ \frac{2}{3}(w \oplus v),$

i.e.,

$$(A(w) \wedge J(w) + I(w)) \oplus (A(v) \wedge J(v) + I(v)) \leq \frac{2}{5}((A(w) \wedge J(w)) \oplus (A(v) \wedge J(v))) + \frac{2}{3}(w \oplus v),$$

Hence, A is J -restricted accretive mapping with constants $\xi_1 = \frac{2}{5}$ and $\xi_2 = \frac{2}{3}$, respectively.

3. Formulation of $GSMOVIP$ and Existence Result

For $i \in \{1,2,3\}$, let B be a real ordered Banach space and K be a normal cone with normal constant δ_K, and let $B \times B \times B$ be an real ordered product Banach space. Let $P_i, Q_i: B \times B \times B \to B$ and $g_i, f_i, h_i: B \to B$ be the ordered single-valued comparison mappings. We consider the generalized system of mixed ordered variational inequalities problems involving $\oplus$ and e operations (in short, GSMOVIP):

For $\phi_1, \phi_2, \phi_3 \in B$, find $(u, v, w) \in B \times B \times B$ such that

$$\left.\begin{aligned} P_1(f_1(u), v, w) + Q_1(u, f_2(v), f_3(w)) \geq \phi_1 \\ P_2(u, g_2(v), w) \oplus Q_2(g_1(u), v, g_3(w)) \geq \phi_2 \\ P_3(u, v, h_3(w)) e Q_3(h_1(u), h_2(v), w) \geq \phi_3 \end{aligned}\right\} \tag{3.1}$$

In addition if $P_1(f_1(u), v, .) = P_1(f_1(u), v), Q_1(u, f_2(v), .) = Q_1(u, v), P_2(u, v, .) = P_2(u, v)$, $Q_2(g_1(u), v, g_3(w)) = Q_2(u, g_3(v)), f_2, f_3, g_1, g_2 = I$ (identity mappings) and $P_3, Q_3 = 0$ (zero mappings), it is clear that for suitable choices of mappings involved in the formulation of problem (4.1), one can obtain many system of variational inequalities problems and variational inequalities studied in recent past [38, 39, 41].

Now, we have the following fixed point formulation of our considered GSMOVIP (3.1).

Lemma 3.1. For $i \in \{1,2,3\}$, let $P_i, Q_i: B \times B \times B \to B$ and $g_i, f_i, h_i: B \to B$ be the ordered single-valued comparison mappings with each other such that P_i is $(\kappa_i, \nu_i, \tau_i)$ -ordered Lipschitz continuous mapping, Q_i is $(\kappa_i', \nu_i', \tau_i')$ -ordered Lipschitz continuous mapping, g_i is μ_{g_i}-ordered compression mapping, f_i is μ_{f_i}-ordered compression mapping and h_i is μ_{h_i}-ordered compression mapping, respectively. Then, the GSMOVIP (4.1) has a solution (u, v, w) if and only if there exist three ordered compressions mappings J_1, J_2 and J_3 such that the vector-valued mapping $\vec{F} = (F_1(u, v, w), F_2(u, v, w), F_3(u, v, w)): B \times B \times B \to B \times B \times B$,

$$\left.\begin{array}{c} F_1(u, v, w) = (P_1(f_1(u), v, w) + Q_1(u, f_2(v), f_3(w)) - \phi_1) \wedge J_1(u) + I(u) \\ F_2(u, v, w) = (P_2(u, g_2(v), w) \oplus Q_2(g_1(u), v, g_3(w)) - \phi_2) \wedge J_2(v) + I(v) \\ F_3(u, v, w) = (P_3(u, v, h_3(w)) e Q_3(h_1(u), h_2(v), w) - \phi_3) \wedge J_3(w) + I(w) \end{array}\right\} \tag{3.2}$$

has the fixed point (u^*, v^*, w^*) in an real ordered product Banach space $B \times B \times B$, where I is identity mapping on B.

Proof. Let (u^*, v^*, w^*) be a fixed point of the vector-valued mapping (4.2). Then, obviously (u^*, v^*, w^*) is a solution of GSMOVIP (4.1). On the other hand, choosing

$$J_1(u) = (0, \quad if\ 0 \leq P_1(f_1(u), v, w) + Q_1(u, f_2(v), f_3(w)) - \phi_1, \varsigma_1 u + \rho_1, \quad otherwise,$$

$$J_2(v) = (0, \quad if\ 0 \leq P_2(u, g_2(v), w) \oplus Q_2(g_1(u), v, g_3(w)) - \phi_2, \varsigma_2 v + \rho_2, \quad otherwise,$$

and

$$J_3(w) = (0, \quad if\ 0 \leq P_3(u, v, h_3(w)) e Q_3(h_1(u), h_2(v), w) - \phi_3, \varsigma_3 w + \rho_3, \quad otherwise,$$

where $\varsigma_1, \varsigma_2, \varsigma_3 \in (0,1)$, and $\rho_1, \rho_2, \rho_3 \in R$, if (u^*, v^*, w^*) is a solution of GSMOVIP (4.1), then

$$\left.\begin{array}{c} (P_1(f_1(u^*), v^*, w^*) + Q_1(u^*, f_2(v^*), f_3(w^*)) - \phi_1) \wedge J_1(u^*) + I(u^*) = u^* \\ (P_2(u^*, g_2(v^*), w^*) \oplus Q_2(g_1(u^*), v^*, g_3(w^*)) - \phi_2) \wedge J_2(v^*) + I(v^*) = v^* \\ (P_3(u^*, v^*, h_3(w^*)) e Q_3(h_1(u^*), h_2(v^*), w^*) - \phi_3) \wedge J_3(w^*) + I(w^*) = w^* \end{array}\right\} \tag{3.3}$$

hold. Therefore, (u^*, v^*, w^*) is a fixed point of the vector-valued mapping (3.2), where the mappings J_1, J_2 and J_3 are ordered compressions. This completes the proof.

4. Main Results

In this section, we discuss the existence and convergence result of the proposed algorithms for GSMOVIP (4.1).

Theorem 4.1. For $i \in \{1,2,3\}$, let $P_i, Q_i: B \times B \times B \to B$ and $g_i, f_i, h_i, J_i: B \to B$ be the ordered single-valued comparison mappings with each other such that P_i is $(\kappa_i, \nu_i, \tau_i)$ -ordered Lipschitz continuous mapping, Q_i is $(\kappa_i', \nu_i', \tau_i')$ -ordered Lipschitz continuous mapping, g_i is μ_{g_i}-ordered compression mapping, f_i is μ_{f_i}-ordered compression mapping, h_i is μ_{h_i}-ordered compression mapping and J_i is μ_{J_i}-ordered compression mapping , respectively. Suppose $P_1 + Q_1 - \phi_1$ is a J_1-restricted-accretive mapping with constants (ξ_1, ξ_2), with respect to first argument, $P_2 \oplus Q_2 - \phi_2$ is a J_2-restricted-accretive mapping with constants (ρ_1, ρ_2), with respect to second argument and $P_3 e Q_3 - \phi_3$ is a J_3-restricted-accretive mapping with constants (σ_1, σ_2), with respect to third argument, respectively. In addition, if $P_i, Q_i, g_i, f_i, h_i, J_i$ are compared to each other, the following condition is satisfied:

$$\left.\begin{array}{c}\delta_K\, max\{\,\xi_1((\kappa_1\mu_{f_1}+\kappa_1^{'})\vee\mu_{J_1}+\xi_2),\xi_1(\nu_1+\nu_1^{'}\mu_{f_2}),\xi_1(\tau_1+\tau_1^{'}\mu_{f_3}),\\ \rho_1(\kappa_2\oplus\kappa_2^{'}\mu_{g_1}),\rho_1((\nu_2\mu_{g_2}\oplus\nu_2^{'})\vee\mu_{J_2}+\rho_2),\rho_1(\tau_2\oplus\tau_2^{'}\mu_{g_3}),\\ \sigma_1(\kappa_3\oplus\kappa_3^{'}\mu_{h_1}),\sigma_1(\nu_3\oplus\nu_3^{'}\mu_{h_2}),\sigma_1((\tau_3\mu_{h_3}\oplus\tau_3^{'})\vee\mu_{J_3}+\sigma_2)\}<1\end{array}\right\} \quad (4.1)$$

holds. then the GSMOVIP (4.1) admits a solution (u^*,v^*,w^*) which is a fixed point of the vector-valued mapping $\vec{F}=(F_1(u,v,w),F_2(u,v,w),F_3(u,v,w))$ in an real ordered product Banach space $B\times B\times B$.

Proof. Let B be a real ordered Banach space, and let $B\times B\times B$ be an ordered product real Banach space. Setting

$$\left.\begin{array}{r}F_1(u,v,w)=(P_1(f_1(u),v,w)+Q_1(u,f_2(v),f_3(w))-\phi_1)\wedge J_1(u)+I(u)\\ F_2(u,v,w)=(P_2(u,g_2(v),w)\oplus Q_2(g_1(u),v,g_3(w))-\phi_2)\wedge J_2(v)+I(v)\\ F_3(u,v,w)=(P_3(u,v,h_3(w))eQ_3(h_1(u),h_2(v),w)-\phi_3)\wedge J_3(w)+I(w)\end{array}\right\} \quad (4.2)$$

Since $P_1+Q_1-\phi_1$ is a J_1-restricted-accretive mapping with (ξ_1,ξ_2), $P_2\oplus Q_2-\phi_2$ is a J_2-restricted-accretive mapping with (ρ_1,ρ_2), and $P_3eQ_3-\phi_3$ is a J_3-restricted-accretive mapping with (σ_1,σ_2), and P_i is (κ_i,ν_i,τ_i) -ordered Lipschitz continuous mapping and Q_i is $(\kappa_i',\nu_i',\tau_i')$ -ordered Lipschitz continuous mapping, respectively. For any given $u_j,v_j,w_j\in B,(j=1,2)$ which are compared to each other, by Lemma 3.1 and Definition 3.1, we can obtain the following inequalities:

$$\begin{aligned}
&0\le F_1(u_1,v_1,w_1)\oplus F_1(u_2,v_2,w_2)\\
&=((P_1(f_1(u_1),v_1,w_1)+Q_1(u_1,f_2(v_1),f_3(w_1))-\phi_1)\wedge J_1(u_1)+I(u_1))\\
&\oplus((P_1(f_1(u_2),v_2,w_2)+Q_1(u_2,f_2(v_2),f_3(w_2))-\phi_1)\wedge J_1(u_2)+I(u_2))\\
&\le\xi_1(((P_1(f_1(u_1),v_1,w_1)+Q_1(u_1,f_2(v_1),f_3(w_1))-\phi_1)\wedge J_1(u_1))\\
&\oplus((P_1(f_1(u_2),v_2,w_2)+Q_1(u_2,f_2(v_2),f_3(w_2))-\phi_1)\wedge J_1(u_2)))+\xi_2(u_1\oplus u_2)\\
&\le\xi_1(((P_1(f_1(u_1),v_1,w_1)+Q_1(u_1,f_2(v_1),f_3(w_1)))\oplus(P_1(f_1(u_2),v_2,w_2)\\
&+Q_1(u_2,f_2(v_2),f_3(w_2))))\vee(J_1(u_1)\oplus J_1(u_2)))+\xi_2(u_1\oplus u_2)\\
&\le\xi_1((P_1(f_1(u_1),v_1,w_1)\oplus P_1(f_1(u_2),v_2,w_2)+Q_1(u_1,f_2(v_1),f_3(w_1))\\
&\oplus Q_1(u_2,f_2(v_2),f_3(w_2)))\vee(\mu_{J_1}(u_1\oplus u_2)))+\xi_2(u_1\oplus u_2)\\
&\le\xi_1((\kappa_1\mu_{f_1}(u_1\oplus u_2)+\nu_1(v_1\oplus v_2)+\tau_1(w_1\oplus w_2))+(\kappa_1^{'}(u_1\oplus u_2)\\
&+\nu_1^{'}\mu_{f_2}(v_1\oplus v_2)+\tau_1^{'}\mu_{f_3}(w_1\oplus w_2)))\vee(\mu_{J_1}(u_1\oplus u_2)))+\xi_2(u_1\oplus u_2)\\
&\le\xi_1(((\kappa_1\mu_{f_1}+\kappa_1^{'})(u_1\oplus u_2)+(\nu_1+\nu_1^{'}\mu_{f_2})(v_1\oplus v_2)\\
&+(\tau_1+\tau_1^{'}\mu_{f_3})(w_1\oplus w_2))\vee(\mu_{J_1}(u_1\oplus u_2)))+\xi_2(u_1\oplus u_2)\\
&\le\xi_1((\kappa_1\mu_{f_1}+\kappa_1^{'})\vee\mu_{J_1}+\xi_2)(u_1\oplus u_2)+\xi_1(\nu_1+\nu_1^{'}\mu_{f_2})(v_1\oplus v_2)\\
&+\xi_1(\tau_1+\tau_1^{'}\mu_{f_3})(w_1\oplus w_2)\\
&\le Y_1(u_1\oplus u_2)+Y_2(v_1\oplus v_2)+Y_3(w_1\oplus w_2),
\end{aligned} \quad (4.3)$$

where $\Upsilon_1=\xi_1((\kappa_1\mu_{f_1}+\kappa_1')\vee\mu_{J_1}+\xi_2),\Upsilon_2=\xi_1(\nu_1+\nu_1'\mu_{f_2})$ and $Y_3=\xi_1(\tau_1+\tau_1^{'}\mu_{f_3})$.

Using the same argument as for (4.3), we calculate

$$\begin{aligned}
&0\le F_2(u_1,v_1,w_1)\oplus F_2(u_2,v_2,w_2)\\
&=((P_2(u_1,g_2(v_1),w_1)\oplus Q_2(g_1(u_1),v_1,g_3(w_1))-\phi_2)\wedge J_2(v_1)+I(v_1))
\end{aligned}$$

$$\oplus\ ((P_2(u_2, g_2(v_2), w_2) \oplus Q_2(g_1(u_2), v_2, g_3(w_2)) - \phi_2) \wedge J_2(v_2) + I(v_2))$$

$$\leq \rho_1(((P_2(u_1, g_2(v_1), w_1) \oplus Q_2(g_1(u_1), v_1, g_3(w_1)) - \phi_2) \wedge J_2(v_2))$$

$$\oplus\ ((P_2(u_2, g_2(v_2), w_2) \oplus Q_2(g_1(u_2), v_2, g_3(w_2)) - \phi_2) \wedge J_2(v_2))) + \rho_2(v_1 \oplus v_2)$$

$$\leq \rho_1(((P_2(u_1, g_2(v_1), w_1) \oplus Q_2(g_1(u_1), v_1, g_3(w_1))) \oplus (P_2(u_2, g_2(v_2), g_2(w_2))$$

$$\oplus\ Q_2(g_1(u_2), v_2, g_3(w_2)))) \vee (J_2(v_1) \oplus J_2(v_2))) + \rho_2(v_1 \oplus v_2)$$

$$\leq \rho_1(((P_1(u_1, g_2(v_1), w_1) \oplus P_1(u_2, g_2(v_2), w_2)) \oplus (Q_1(g_1(u_1), v_1, g_3(w_1))$$

$$\oplus\ Q_2(g_1(u_2), v_2, g_3(w_2)))) \vee (\mu_{J_2}(v_1 \oplus v_2))) + \rho_2(v_1 \oplus v_2)$$

$$\leq \rho_1(((\kappa_2(u_1 \oplus u_2) + \nu_2\mu_{g_2}(v_1 \oplus v_2) + \tau_2(w_1 \oplus w_2)) \oplus (\kappa_2^{'}\mu_{g_1}(u_1 \oplus u_2)$$

$$+\nu_2^{'}(v_1 \oplus v_2) + \tau_2^{'}\mu_{g_3}(w_1 \oplus w_2))) \vee (\mu_{J_2}(v_1 \oplus v_2))) + \rho_2(v_1 \oplus v_2)$$

$$\leq \rho_1(((\kappa_2 \oplus \kappa_2^{'}\mu_{g_1})(u_1 \oplus u_2) + (\nu_2\mu_{g_2} \oplus \nu_2^{'})(v_1 \oplus v_2)$$

$$+(\tau_2 \oplus \tau_2^{'}\mu_{g_3})(w_1 \oplus w_2)) \vee (\mu_{J_2}(v_1 \oplus v_2))) + \rho_2(v_1 \oplus v_2)$$

$$\leq \rho_1(\kappa_2 \oplus \kappa_2^{'}\mu_{g_1})(u_1 \oplus u_2) + \rho_1((\nu_2\mu_{g_2} \oplus \nu_2^{'}) \vee \mu_{J_2} + \rho_2)(v_1 \oplus v_2)$$

$$+\rho_1(\tau_2 \oplus \tau_2^{'}\mu_{g_3})(w_1 \oplus w_2)$$

$$\leq \Psi_1(u_1 \oplus u_2) + \Psi_2(v_1 \oplus v_2) + \Psi_3(w_1 \oplus w_2), \tag{4.4}$$

where $\Psi_1 = \rho_1(\kappa_2 \oplus \kappa_2'\mu_{g_1}), \Psi_2 = \rho_1((\nu_2\mu_{g_2} \oplus \nu_2') \vee \mu_{J_2} + \rho_2)$ and $\Psi_3 = \rho_1(\tau_2 \oplus \tau_2^{'}\mu_{g_3})$.

Using the same argument as for (4.3), we calculate

$$0 \leq F_3(u_1, v_1, w_1) \oplus F_3(u_2, v_2, w_2)$$

$$= ((P_3(u_1, v_1, h_3(w_1))eQ_3(h_1(u_1), h_2(v_1), w_1) - \phi_3) \wedge J_3(w_1) + I(w_1))$$

$$\oplus\ ((P_3(u_2, v_2, h_3(w_2))eQ_3(h_1(u_2), h_2(v_2), w_2) - \phi_3) \wedge J_3(w_2) + I(w_2))$$

$$\leq \sigma_1(((P_3(u_1, v_1, h_3(w_1))eQ_3(h_1(u_1), h_2(v_1), w_1) - \phi_3) \wedge J_3(w_2))$$

$$\oplus\ ((P_3(u_2, v_2, h_3(w_2))eQ_3(h_1(u_2), h_2(v_2), w_2) - \phi_3) \wedge J_3(w_2))) + \sigma_2(w_1 \oplus w_2)$$

$$\leq \sigma_1(((P_3(u_1, v_1, h_3(w_1))eQ_3(h_1(u_1), h_2(v_1), w_1)) \oplus (P_3(u_2, v_2, h_2(w_2))$$

$$eQ_3(h_1(u_2), h_2(v_2), w_2))) \vee (J_3(w_1) \oplus J_3(w_2))) + \sigma_2(w_1 \oplus w_2)$$

$$\leq \sigma_1((|-1|((P_3(u_1, v_1, h_3(w_1)) \oplus Q_3(h_1(u_1), h_2(v_1), w_1)) \oplus (P_3(u_2, v_2, h_2(w_2))$$

$$\oplus\ Q_3(h_1(u_2), h_2(v_2), w_2)))) \vee (J_3(w_1) \oplus J_3(w_2))) + \sigma_2(w_1 \oplus w_2)$$

$$\leq \sigma_1(((P_3(u_1, v_1, h_3(w_1)) \oplus P_3(u_2, v_2, h_3(w_2))) \oplus (Q_3(h_1(u_1), h_2(v_1), w_1)$$

$$\oplus\ Q_3(h_1(u_2), h_2(v_2), w_2))) \vee (\mu_{J_3}(w_1 \oplus w_2))) + \sigma_2(w_1 \oplus w_2)$$

$$\leq \sigma_1(((\kappa_3(u_1 \oplus u_2) + \nu_3(v_1 \oplus v_2) + \tau_3\mu_{h_3}(w_1 \oplus w_2)) \oplus (\kappa_3^{'}\mu_{h_1}(u_1 \oplus u_2)$$

$$+\nu_3^{'}\mu_{h_2}(v_1 \oplus v_2) + \tau_3^{'}(w_1 \oplus w_2))) \vee (\mu_{J_3}(w_1 \oplus w_2))) + \sigma_2(w_1 \oplus w_2)$$

$$\leq \sigma_1(((\kappa_3 \oplus \kappa_3^{'}\mu_{h_1})(u_1 \oplus u_2) + (\nu_3 \oplus \nu_3^{'}\mu_{h_2})(v_1 \oplus v_2)$$

$$+(\tau_3\mu_{h_3} \oplus \tau_3^{'})(w_1 \oplus w_2)) \vee (\mu_{J_3}(w_1 \oplus w_2))) + \sigma_2(w_1 \oplus w_2)$$

$$\leq \sigma_1(\kappa_3 \oplus \kappa_3^{'}\mu_{h_1})(u_1 \oplus u_2) + \sigma_1((\nu_3 \oplus \nu_3^{'}\mu_{h_2})(v_1 \oplus v_2)$$

$$+\sigma_1((\tau_3\mu_{h_3} \oplus \tau_3^{'}) \vee \mu_{J_3} + \sigma_2)(w_1 \oplus w_2)$$

$$\leq \Omega_1(u_1 \oplus u_2) + \Omega_2(v_1 \oplus v_2) + \Omega_3(w_1 \oplus w_2). \tag{4.5}$$

where $\Omega_1 = \sigma_1(\kappa_3 \oplus \kappa_3'\mu_{h_1}), \Omega_2 = \sigma_1((\nu_3 \oplus \nu_3'\mu_{h_2})$ and $\Omega_3 = \sigma_1((\tau_3\mu_{h_3} \oplus \tau_3^{'}) \vee \mu_{J_3} + \sigma_2)$.

Combining (4.3), (4.4) and (4.5), we have

$$0 \leq \vec{F}(u_1, v_1, w_1) \oplus \vec{F}(u_2, v_2, w_2) = (F_1, F_2, F_3)(u_1, v_1, w_1) \oplus (F_1, F_2, F_3)(u_2, v_2, w_2) \leq \Phi((u_1, v_1, w_1) \oplus (u_2, v_2, w_2)), \tag{4.6}$$

where

$$\Phi = \begin{pmatrix} \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Omega_1 & \Omega_2 & \Omega_3 \end{pmatrix}$$

By Definition 2.1 (i), we have

$$\begin{aligned} \|\vec{F}(u_1, v_1, w_1) \oplus \vec{F}(u_2, v_2, w_2)\| &= \|(F_1, F_2, F_3)(u_1, v_1, w_1) \oplus (F_1, F_2, F_3)(u_2, v_2, w_2)\| \\ &\leq \delta_K \|\Phi\| \, \|((u_1, v_1, w_1) \oplus (u_2, v_2, w_2))\|, \end{aligned} \tag{4.7}$$

where $\|\Phi\| = max\{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Psi_1, \Psi_2, \Psi_3, \Omega_1, \Omega_2, \Omega_3\}$ and δ_K is a normal constant of K. It follows from (4.7) and the assumption condition (4.1) that $0 < \delta_K P\Phi P < 1$, and hence the vector-valued mapping

$$(F_1, F_2, F_3)^T = ((P_1(f_1(.), ., .) + Q_1(., f_2(.), f_3(.)) - \phi_1) \wedge J_1(.) + I(.),$$

$$(P_2(., g_2(.), .) \oplus Q_2(g_1(.), ., g_3(.)) - \phi_2) \wedge J_2(.) + I(.),$$

$$(P_3(., ., h_3(.))eQ_3(h_1(.), h_2(.), .) - \phi_3) \wedge J_3(.) + I(.))^T$$

has a fixed point (u^*, v^*, w^*) for Lemma 4.1, in an ordered product Banach space $B \times B \times B$, which is a solution for GSMOVIP (4.1) by Lemma 4.1. this completes the proof.

Theorem 4.2. Suppose all the mappings P_i, Q_i, g_i, f_i, h_i and J_i are similar as in Theorem 4.1 such that all the hypotheses of Theorem 4.1 are satisfied. Besides, admit that the following assumptions hold:

$$\left.\begin{aligned} &max\{\xi_1((\kappa_1\mu_{f_1} + \kappa_1^{'}) \vee \mu_{J_1} + \xi_2), \xi_1(\nu_1 + \nu_1^{'}\mu_{f_2}), \xi_1(\tau_1 + \tau_1^{'}\mu_{f_3}), \\ &\rho_1(\kappa_2 \oplus \kappa_2^{'}\mu_{g_1}), \rho_1((\nu_2\mu_{g_2} \oplus \nu_2^{'}) \vee \mu_{J_2} + \rho_2), \rho_1(\tau_2 \oplus \tau_2^{'}\mu_{g_3}), \sigma_1(\kappa_3 \oplus \kappa_3^{'}\mu_{h_1}), \\ &\sigma_1(\nu_3 \oplus \nu_3^{'}\mu_{h_2}), \sigma_1((\tau_3\mu_{h_3} \oplus \tau_3^{'}) \vee \mu_{J_3} + \sigma_2)\} < min\{\frac{1}{\delta_K}, 1\} \end{aligned}\right\} \tag{4.8}$$

Then the iterative sequences $\{(u_n, v_n, w_n)\}$ generated by the following algorithm:

$$\left.\begin{aligned} u_{n+1} &= (1-\alpha)u_n + \alpha(P_1(f_1(u_n), v_n, w_n) + Q_1(u_n, f_2(v_n), f_3(w_n)) - \phi_1) \wedge J_1(u_n) \\ &\quad + I(u_n) \\ v_{n+1} &= (1-\beta)v_n + \beta(P_2(u_n, g_2(v_n), w_n) \oplus Q_2(g_1(u_n), v_n, g_3(w_n)) - \phi_2) \wedge J_2(v_n) \\ &\quad + I(v_n) \\ w_{n+1} &= (1-\gamma)w_n + \gamma(P_3(u_n, v_n, h_3(w_n))eQ_3(h_1(u_n), h_2(v_n), w_n) - \phi_3) \wedge J_3(w_n) \\ &\quad + I(w_n) \end{aligned}\right\} \tag{4.9}$$

for any $u_0, v_0, w_0 \in B, u_0 \propto u_1, v_0 \propto v_1, w_0 \propto w_1, (u_0, v_0, w_0) \propto (u_1, v_1, w_1)$ and $0 < \alpha, \beta, \gamma < 1$, converges strongly to (u^*, v^*, w^*), which is a solution of GSMOVIP (4.1).

Proof. Let the assumption conditions in Theorem 4.1 hold. For any given $u_0, v_0, w_0 \in B$, and $u_0 \propto u_1, v_0 \propto v_1, w_0 \propto w_1, (u_0, v_0, w_0) \propto (u_1, v_1, w_1)$, setting

$$\left.\begin{aligned} F_1(u,v,w) = (P_1(f_1(u),v,w) + Q_1(u,f_2(v),f_3(w)) - \phi_1) \wedge J_1(u) + I(u) \\ F_2(u,v,w) = (P_2(u,g_2(v),w) \oplus Q_2(g_1(u),v,g_3(w)) - \phi_2) \wedge J_2(v) + I(v) \\ F_3(u,v,w) = (P_3(u,v,h_3(w))eQ_3(h_1(u),h_2(v),w) - \phi_3) \wedge J_3(w) + I(w), \end{aligned}\right\} \tag{4.10}$$

then for any $0 < \alpha, \beta, \gamma < 1$, by algorithm (4.9), and (4.3)-(4.5), we have

$$\begin{aligned} &0 \leq u_{n+1} \oplus u_n \\ &= [(1-\alpha)u_n + \alpha F_1(u_n,v_n,w_n)] \oplus [(1-\alpha)u_{n-1} + \alpha F_1(u_{n-1},v_{n-1},w_{n-1})] \\ &\leq (1-\alpha)(u_n \oplus u_{n-1}) + \alpha(F_1(u_n,v_n,w_n) \oplus F_1(u_{n-1},v_{n-1},w_{n-1})) \\ &\leq (1-\alpha)(u_n \oplus u_{n-1}) + \alpha(Y_1(u_n \oplus u_{n-1}) + Y_2(v_n \oplus v_{n-1}) + Y_3(w_n \oplus w_{n-1})) \\ &\leq (1-\alpha(1-Y_1))(u_n \oplus u_{n-1}) + \alpha Y_2(v_n \oplus v_{n-1}) + Y_3(w_n \oplus w_{n-1}). \end{aligned} \tag{4.11}$$

In similar, we have

$$\begin{aligned} &0 \leq v_{n+1} \oplus v_n \\ &= [(1-\beta)v_n + \beta F_2(u_n,v_n,w_n)] \oplus [(1-\beta)v_{n-1} + \beta F_2(u_{n-1},v_{n-1},w_{n-1})] \\ &\leq (1-\beta)(v_n \oplus v_{n-1}) + \beta(F_2(u_n,v_n,w_n) \oplus F_2(u_{n-1},v_{n-1},w_{n-1})) \\ &\leq (1-\beta)(v_n \oplus v_{n-1}) + \beta(\Psi_1(u_n \oplus u_{n-1}) + \Psi_2(v_n \oplus v_{n-1}) + \Psi_3(w_n \oplus w_{n-1})) \\ &\leq \beta\Psi_1(u_n \oplus u_{n-1}) + (1-\beta(1-\Psi_2))(v_n \oplus v_{n-1}) + \Psi_3(w_n \oplus w_{n-1}). \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &0 \leq w_{n+1} \oplus w_n \\ &= [(1-\gamma)w_n + \beta F_3(u_n,v_n,w_n)] \oplus [(1-\gamma)w_{n-1} + \gamma F_3(u_{n-1},v_{n-1},w_{n-1})] \\ &\leq (1-\gamma)(w_n \oplus w_{n-1}) + \gamma(F_3(u_n,v_n,w_n) \oplus F_3(u_{n-1},v_{n-1},w_{n-1})) \\ &\leq (1-\gamma)(w_n \oplus w_{n-1}) + \gamma(\Omega_1(u_n \oplus u_{n-1}) + \Omega_2(v_n \oplus v_{n-1}) + \Omega_3(w_n \oplus w_{n-1})) \\ &\leq \gamma\Omega_1(u_n \oplus u_{n-1}) + \gamma\Omega_2(v_n \oplus v_{n-1}) + (1-\gamma(1-\Omega_3))(w_n \oplus w_{n-1}). \end{aligned} \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we have

$$(u_{n+1},v_{n+1},w_{n+1}) \oplus (u_n,v_n,w_n) \leq \Gamma((u_n,v_n,w_n) \oplus (u_{n-1},v_{n-1},w_{n-1})),$$

where

$$\Gamma = \begin{pmatrix} 1-\alpha(1-Y_1) & \alpha Y_2 & \alpha Y_3 \\ \beta\Psi_1 & 1-\beta(1-\Psi_2) & \beta\Psi_3 \\ \gamma\Omega_1 & \gamma\Omega_2 & 1-\gamma(1-\Omega_3) \end{pmatrix}$$

By Definition 3.1 (i), we have

$$P(u_{n+1},v_{n+1},w_{n+1}) \oplus (u_n,v_n,w_n)P \leq \delta_K P\Gamma P P(u_n,v_n,w_n) \oplus (u_{n-1},v_{n-1},w_{n-1})P,$$

where $P\Gamma P = max\{1-\alpha(1-\Upsilon_1), \alpha\Upsilon_2, \alpha\Upsilon_3, \beta\Psi_1, 1-\beta(1-\Psi_2), \beta\Psi_3, \gamma\Omega_1, \gamma\Omega_2, 1-\gamma(1-\Omega_3)\}$ and δ_K is a normal constant of K. It follows from (4.14) and the assumption condition (4.8) that $\delta_K P\Gamma P < 1$ is true. Hence the sequence $(u_n,v_n,w_n)^T \to (u^*,v^*,w^*)$ is strongly converges. Since P_i, Q_i, g_i, f_i, h_i and J_i are ordered compressions, and they are comparisons of each other, so that

$$\left.\begin{aligned} (P_1(f_1(u^*),v^*,w^*) + Q_1(u^*,f_2(v^*),f_3(w^*)) - \phi_1) \wedge J_1(u^*) + I(u^*) = u^* \\ (P_2(u^*,g_2(v^*),w^*) \oplus Q_2(g_1(u^*),v^*,g_3(w^*)) - \phi_2) \wedge J_2(v^*) + I(v^*) = v^* \\ (P_3(u^*,v^*,h_3(w^*))eQ_3(h_1(u^*),h_2(v^*),w^*) - \phi_3) \wedge J_3(w^*) + I(w^*) = w^* \end{aligned}\right\} \tag{4.14}$$

hold. Therefore, (u^*, v^*, w^*) is a fixed point of the vector-valued mapping

$$(F_1, F_2, F_3)^T = ((P_1(f_1(.),.,.) + Q_1(.,f_2(.),f_3(.)) - \phi_1) \wedge J_1(.) + I(.),$$
$$(P_2(.,g_2(.),.) \oplus Q_2(g_1(.),.,g_3(.)) - \phi_2) \wedge J_2(.) + I(.),$$
$$(P_3(.,.,h_3(.))eQ_3(h_1(.),h_2(.),.) - \phi_3) \wedge J_3(.) + I(.))^T$$

in an ordered product Banach space $B \times B \times B$, which is a solution for GSMOVIP (4.1) by Lemma 4.1. This completes the proof.

The following numerical example gives the guarantee that all the proposed conditions of Theorem 4.1 are satisfied.

Example 4.1. For each $i \in \{1,2,3\}$, and let $B = R$, with the usual inner product and norm and $K = \{x \in H_p: 0 \le u \le 1\}$ be a normal cone with normal constant $\delta_K = 1$. Let $g_i, f_i, h_i, J_i: B \to B$ be the mappings defined by for all $u, v, w \in B$

$$f_1(u) = \frac{u}{30}, f_2(v) = \frac{v}{20}, f_3(w) = \frac{w}{40}, g_1(u) = \frac{u}{40}, g_2(v) = \frac{v}{30}, g_3(w) = \frac{w}{40}, h_1(u) = \frac{u}{10},$$

$$h_2(v) = -\frac{v}{10} + \frac{1}{10}, \quad h_3(w) = \frac{3w}{50}, \quad J_1(u) = \frac{u}{12} - \frac{1}{24}, \quad J_2(v) = \frac{3v-1}{45}, \quad J_3(w) = \frac{w}{20} - \frac{1}{10}.$$

Suppose that the mappings $P_i: B \times B \times B \to B$ are defined by

$$P_1(f_1(u), v, w) = \frac{3}{4} f_1(u) + \frac{v}{20} + \frac{w}{5}, \quad P_2(u, g_2(v), w) = \frac{u}{20} + \frac{1}{2} g_2(v) + \frac{w}{30}, and$$

$$P_3(u, v, h_3(w)) = \frac{u + 2v}{50} - \frac{1}{3} h_3(w), \quad \forall\, u, v, w \in B,$$

and the mappings $Q_i: B \times B \times B \to B$ are defined by

$$Q_1(u, f_2(v), f_3(w)) = \frac{u}{40} - f_2(v) - 8f_3(w), Q_2(g_1(u), v, g_3(w)) = 2g_1(u) + \frac{v}{15} + \frac{4}{3} g_3(w),$$

$$and \quad Q_3(h_1(u), h_2(v), w) = \frac{1}{5} h_1(u) - \frac{2}{5} h_2(v) + \frac{w}{50} + \frac{1}{25}, \forall\, u, v, w \in B.$$

Now,

$$J_1(u_1) \oplus J_1(u_2) = (\frac{u_1}{12} - \frac{1}{24}) \oplus (\frac{u_2}{12} - \frac{1}{24}) \le (\frac{u_1}{12} \oplus \frac{u_2}{12}) + (\frac{1}{24} \oplus \frac{1}{24}) = \frac{1}{12}(u_1 \oplus u_2) \le \frac{1}{10}(u_1 \oplus u_2),$$

i.e.,

$$J_1(u_1) \oplus J_1(u_2) \le \frac{1}{10}(u_1 \oplus u_2).$$

Hence, J_1 is $\frac{1}{10}$ -ordered compression mapping. In the similar way, it is easy to verify that f_1 is $\frac{1}{25}$ -ordered compression, f_2 is $\frac{1}{10}$- ordered compression, f_3 is $\frac{1}{30}$- ordered compression, g_1 is $\frac{1}{30}$- ordered compression, g_2 is $\frac{1}{20}$ -ordered compression, g_3 is $\frac{1}{35}$ -ordered compression, h_1 is $\frac{1}{9}$ -ordered compression, h_2 is $\frac{1}{8}$ -ordered compression, h_3 is $\frac{2}{25}$ -ordered compression, J_1 is $\frac{1}{10}$ -ordered compression, J_2 is $\frac{1}{9}$ -ordered compression and J_3 is $\frac{1}{10}$ -ordered compression mappings, respectively. In particular for $\phi_1 = \frac{1}{40}, \phi_2 = \frac{1}{60}$, and $\phi_3 = -\frac{1}{125}$, we obtain

$$F_1(u, v, w) = (P_1(f_1(u), v, w) + Q_1(u, f_2(v), f_3(w)) - \phi_1) \wedge J_1(u) + I(u)$$
$$= ((\tfrac{3}{4} f_1(u) + \tfrac{v}{20} + \tfrac{w}{5}) + (\tfrac{u}{40} - f_2(v) - 8f_3(w)) - \tfrac{1}{40}) \wedge (\tfrac{u}{12} - \tfrac{1}{24}) + u$$

$$= ((\frac{u}{40} + \frac{v}{20} + \frac{w}{5}) + (\frac{u}{40} - \frac{v}{20} - \frac{w}{5}) - \frac{1}{40}) \wedge (\frac{u}{12} - \frac{1}{24}) + u$$

$$= ((\frac{u}{20} - \frac{1}{40}) \wedge (\frac{u}{12} - \frac{1}{24})) + u = \frac{21}{20}u - \frac{1}{40},$$

$$F_2(u,v,w) = (P_2(u, g_2(v), w) \oplus Q_2(g_1(u), v, g_3(w)) - \phi_2) \wedge J_2(v) + I(v)$$

$$= ((\frac{u}{20} + \frac{1}{2}g_2(v) + \frac{w}{30}) \oplus (2g_1(u) + \frac{v}{15} + \frac{4}{3}g_3(w))) \wedge (\frac{3v-1}{45}) + v$$

$$= ((\frac{u}{20} + \frac{v}{60} + \frac{w}{30}) \oplus (\frac{u}{20} + \frac{v}{15} + \frac{w}{30}) - \frac{1}{60}) \wedge (\frac{3v-1}{45}) + v$$

$$= ((\frac{v}{20} - \frac{1}{60}) \wedge (\frac{3v-1}{45})) + v = (\frac{21}{20}v - \frac{1}{60}),$$

$$F_3(u,v,w) = (P_3(u, v, h_3(w)) e Q_3(h_1(u), h_2(v), w) - \phi_3) \wedge J_3(w) + I(w)$$

$$= ((\frac{u+2v}{50} - \frac{w}{50}) e (\frac{u}{50} + \frac{2v}{50} - \frac{2}{50} + \frac{w}{50} + \frac{1}{25}) + \frac{1}{125}) \wedge (\frac{v}{15} - \frac{1}{75}) + v$$

$$= ((-\frac{w}{25} + \frac{1}{125}) \wedge (\frac{v}{15} - \frac{1}{75})) + w$$

$$= \frac{24}{25}w + \frac{1}{125}.$$

Suppose $u_1, v_1, w_1, u_2, v_2, w_2 \in B, u_1 \propto u_2, v_1 \propto v_2$, and $w_1 \propto w_2$, we calculate

$$P_1(f_1(u_1), v_1, w_1) \oplus P_1(f_1(u_2), v_2, w_2) = (\frac{3}{4}f_1(u_1) + \frac{v_1}{20} + \frac{w_1}{5}) \oplus (\frac{3}{4}f_1(u_2) + \frac{v_2}{20} + \frac{w_2}{5})$$

$$= (\frac{u_1}{40} + \frac{v_1}{20} + \frac{w_1}{5}) \oplus (\frac{u_2}{40} + \frac{v_2}{20} + \frac{w_2}{5}) \leq (\frac{u_1}{40} \oplus \frac{u_2}{40}) + (\frac{v_1}{20} \oplus \frac{v_2}{20}) + (\frac{w_1}{5} \oplus \frac{w_2}{5})$$

$$\leq \frac{1}{40}(u_1 \oplus u_2) + \frac{1}{10}(v_1 \oplus v_2) + \frac{1}{5}(w_1 \oplus w_2).$$

Hence, P_1 is $(\frac{1}{40}, \frac{1}{10}, \frac{1}{5})$ -ordered Lipschitz continuous mappings.

In the similar way, it is easy to verify that P_2 is $(\frac{1}{10}, \frac{1}{30}, \frac{1}{15})$ -ordered Lipschitz continuous mappings, P_3 is $(\frac{1}{40}, \frac{1}{20}, \frac{1}{45})$ -ordered Lipschitz continuous mappings, Q_1 is $(\frac{1}{30}, \frac{1}{15}, \frac{1}{4})$ -ordered Lipschitz continuous mappings, Q_2 is $(\frac{1}{15}, \frac{1}{10}, \frac{1}{4})$ -ordered Lipschitz continuous mappings, and Q_3 is $(\frac{1}{45}, \frac{1}{20}, \frac{1}{40})$ -ordered Lipschitz continuous mappings, respectively. Also, we can verify that $P_1 + Q_1 - \phi_1$ is J_1-restricted-accretive mapping with constatnts $(\frac{1}{2}, \frac{3}{5})$, with respect to first argument, $P_2 \oplus Q_2 - \phi_2$ is J_2-restricted-accretive mapping with constatnts $(\frac{3}{5}, \frac{1}{2})$, with respect to second argument, and $P_3 e Q_3 - \phi_3$ is J_3-restricted-accretive mapping with constatnts $(\frac{1}{2}, \frac{1}{2})$, with respect to third argument, respectively. It is also confirmed that assumption (4.1) is satisfied. So, all the conditions of Theorem 4.1 are fulfilled. Therefore, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ is a fixed point of the vector-valued mapping $\vec{F} = (F_1(.), F_2(.), F_3(.))$.By Lemma 4.1, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ is a solution of GSMOVIP (4.1). It is also verified that condition (4.8) is satisfied. Thus, all the assumptions of Theorem 4.2 are fulfilled.

Let $\alpha = \frac{1}{3}, \beta = \frac{1}{2}$ and $\gamma_n = \frac{2}{3}$. Now, we can estimate the sequence $\{(u_n, v_n, w_n)\}$ by the following schemes:

$$u_{n+1} = \frac{61}{60}u_n - \frac{1}{120}$$

$$v_{n+1} = \frac{41}{40}v_n - \frac{1}{32}$$

$$w_{n+1} = \frac{73}{75}w_n - \frac{2}{375}$$

It is also verified that condition (4.8) is satisfied. Thus, all the assumptions of Theorem 4.2 are fulfilled. Hence, the sequence $\{(u_n, v_n, w_n)\}$ converges strongly to the unique solution $(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ of the GSMOVIP (4.1).

All codes are written in MATLAB version $R2019a$, we have the following different initial values $(u_0, v_0, w_0) = (3.5, 4.5, 5.5)$ and $(u_0, v_0, w_0) = (-4.5, -5.5, -6.5)$ which shows that the sequence $\{(u_n, v_n, w_n)\}$ converge to $(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ (Table **1**, Fig. **1-2**).

Table 1: The values of $\{(u_n, v_n, w_n)\}$ with initial values $(u_0, v_0, w_0) = (3.5, 4.5, 5.5)$ and $(u_0, v_0, w_0) = (-4.5, -5.5, -6.5)$.

No. of Iteration (n)	For $(u_0, v_0, w_0) = (3.5, 4.5, 5.5)$			$(u_0, v_0, w_0) = (-4.5, -5.5, -6.5)$		
	u_n	v_n	w_n	u_n	v_n	w_n
1	3.5000	4.5000	5.5000	-4.500	-5.5000	-6.5000
2	2.0750	2.5200	2.9820	-2.1250	–2.7300	-3.3180
3	1.3268	1.4805	1.6600	-0.8781	-1.2757	-1.6474
4	0.9341	0.9347	0.9660	-0.2235	-0.5122	-0.7704
5	0.7279	0.6482	0.6016	0.1201	-0.1114	-0.3099
6	0.6196	0.4978	0.4103	0.3005	0.0989	-0.0682
7	0.5628	0.4188	0.3099	0.3953	0.2094	0.0586
8	0.5329	0.3774	0.2572	0.4450	0.2674	0.1253
9	0.5173	0.3556	0.2295	0.4711	0.2979	0.1602
10	0.5090	0.3442	0.2150	0.4848	0.3139	0.1786
11	0.5047	0.3382	0.2073	0.4920	0.3223	0.1882
12	0.5025	0.3350	0.2033	0.4958	0.3267	0.1933
13	0.5013	0.3334	0.2012	0.4978	0.3290	0.1960
15	0.5003	0.3320	0.2008	0.4993	0.3308	0.1981
17	0.5009	0.3318	0.2005	0.4998	0.3313	0.1987
20	0.5005	0.3323	0.2002	0.4997	0.3321	0.1989
23	0.5001	0.3332	0.2001	0.5000	0.3329	0.1999
25	0.5000	0.3333	0.2000	0.5000	0.3333	0.2000

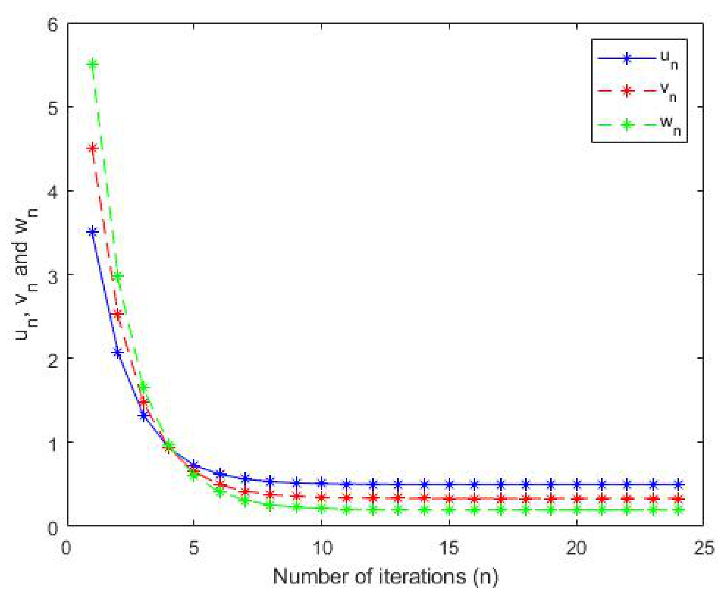

Figure 1: The convergence of $\{(u_n, v_n, w_n)\}$ with initial values $(u_0, v_0, w_0) = (-4.5, -5.5, -6.5)$.

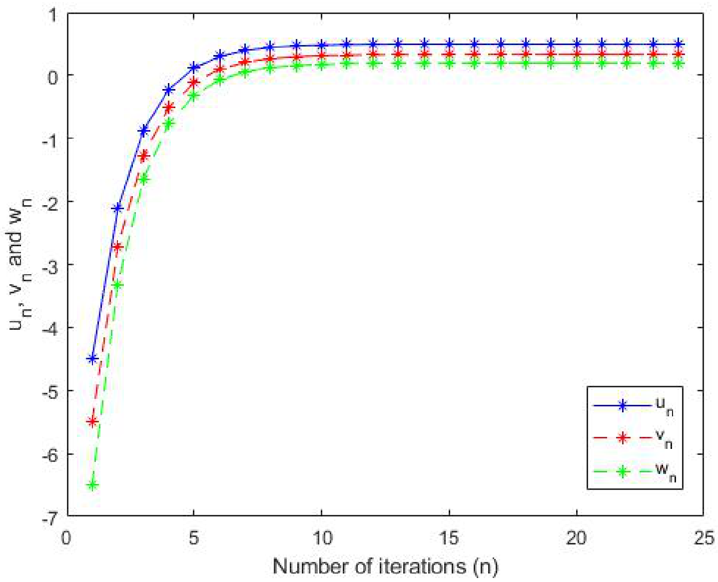

Figure 2: The convergence of $\{(u_n, v_n, w_n)\}$ with initial values $(u_0, v_0, w_0) = (-4.5, -5.5, -6.5)$.

5. Conclusion

In this article, we studied and analyzed a system of mixed ordered variational inequality problems involving XOR and XNOR operations in a real ordered product Banach space and discussed the existence of the solution of our proposed problem. We discussed the convergence criteria of the iterative sequences which assumes that the suggested algorithm converges to the solution of our considered problem. Finally, we demonstrate a numerical example that satisfies all the conditions and show the convergence of the proposed algorithm of our main result. We remark that our results may be solved by the forward-backward splitting method based on the inertial technique with XOR and XNOR operations techniques and other higher dimension spaces.

Conflict of Interest

The authors declare that they have no competing interests.

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Availability of Data and Materials

Not applicable

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