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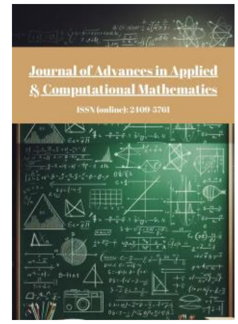

$W^{2,p}$-Regularity of L^p Viscosity Solutions to Fully Nonlinear Elliptic Equations with Low-Order Terms

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ABSTRACT

In this paper, we consider the following fully nonlinear elliptic equation

$$F(D^2u, Du, x) = f(x),$$

where the operator F satisfies structure condition and the gradient of solution has L^p_{loc} growth rate particularly. We employ the technique from geometric tangential analysis whose basic principle is to transfer the good regularity of the recession operator to the original F by approximation methods and establish a prior local $W^{2,p}$ estimates for - L^p-viscosity solutions to the above equation.

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1. Introduction and Main Results

In this paper, we are concerned with the interior $W^{2,p}$ regularity for L^p viscosity solutions of the following equation

$$F(D^2u, Du, x) = f \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\Omega \subseteq R^d$ is a bounded domain, $f \in L^p$, $p > d$ and the operator $F: S(d) \times R^d \times \Omega \to R$ satisfies structure condition, *i.e.*, there exists $b \geq 0$, $b \in L^p_{\text{loc}}(R^d)$ such that

$$M^-(M-N) - b(x)|P-Q| \leq F(M,P,x) - F(N,Q,x) \leq M^+(M-N) + b(x)|P-Q|. \tag{1.2}$$

Above $S(d)$ denotes the set of $d \times d$ real symmetric matrices and M^+ and M^- are Pucci's extremal operators. The celebrated work on the interior $W^{2,p}$ estimates for $p > d$ was done by Caffarelli [1], where he assumed the homogeneous equation driven by the fixed-coefficients counterpart of the operator to have $C^{1,1}$-estimates. This essential assumption is a convexity-like condition on the operator F. $W^{2,p}$ estimates have been generalized by Escauriaza [2] allowing $f \in L^p$ for $p > n - \varepsilon$ for some $\varepsilon > 0$ depending on the ellipticity of the equation which could be traced back to [3]. In [4], Fok obtained the ABP maximum principle for $p > n - \varepsilon_0$. In 1997, S'wiech [5] gave the $W^{2,p}$ estimates for L^p-viscosity solutions for $p > n - \varepsilon_0$ and after that, Li and Zhang [6] also gave the same result by using the different method. Furthermore, Winter [7] obtained the global estimates analogous to interior estimates in [1]. In [8, 9], Krylov established the existence of solutions of the Dirichlet problem for fully nonlinear elliptic equations under relaxed convexity and no convexity assumptions, respectively. Concerning regularity estimates in Hölder spaces, we can refer to papers [10-13].

The challenging question for fully nonlinear elliptic equations with nonconvex structures is whether $W^{2,p}$ regularity is valid. In [14], E.A Pimentel and E.V. Teixeira studied the following equation:

$$F(D^2u) = f \quad \text{with} f \in L^p, p > d.$$

They denote the recession operator associated with F by $F^*(M) = \lim_{\mu \to 0} \mu F(\mu^{-1}M)$. Under the assumption that F^* is convex, they established sharp Sobolev estimates for solutions of the above fully nonlinear elliptic equation. Recently, $W^{2,BMO}$-regularity results were proved by Huang [15], da Silva and Ricarte [16].

In this article, we establish the interior $W^{2,p}$ estimates for the L^p viscosity solutions of (1.1) via recession strategy which is inspired by the above works. Significantly, the assumption of the growth rate of the gradient of the solution in the original structural condition is relaxed by an $L^p_{\text{loc}}(R^d)$ integrable function in our paper. In order to state our main results, we begin by presenting some definitions, notations and assumptions. Firstly, we redefine the recession operator. For $\mu > 0$, set

$$F_\mu(M, 0, x) = \mu F(\mu^{-1}M, 0, x). \tag{1.3}$$

The recession function associated with F is defined by

$$F^*(M,0,x) := \lim_{\mu \to 0} F_\mu(M,0,x). \tag{1.4}$$

An interesting observation concerns the characteristic of F^* which is the same as F. If F satisfies (1.2), so is F^*. Indeed, for given $M, N \in S(d)$, with $N \geq 0$, we have

$$\begin{aligned} F^*(M+N,0,x) - F^*(M,0,x) &= \lim_{\mu \to 0} \mu F\left(\mu^{-1}(M+N),0,x\right) - \mu F\left(\mu^{-1}M,0,x\right) \\ &\leq \lim_{\mu \to 0} \mathrm{M}^+(M-N) \\ &= \mathrm{M}^+(M-N). \end{aligned} \tag{1.5}$$

The remaining inequality follows similarly.

For a fixed $x_0 \in \Omega$, we define the quantity:

$$\beta_{F^*}(x, x_0) := \sup_{X \in S(d)} \frac{\left|F^*(X,0,x) - F^*(X,0,x_0)\right|}{\|X\| + 1},$$

which measures the oscillation of the coefficients of F^* around x_0. For simplicity we often write $\beta_{F^*}(x,0) = \beta_{F^*}(x)$.

We provide two hypotheses for the recession operator as follows. One is that continuous coefficients derived from F^* in the L^p-avarege sense is controlled by small constant.

Assumption 1.1: There exist universal constants $\alpha \in (0,1)$, $\theta_0 > 0$ and $0 < r_0 \ll 1$ such that

$$\left(⨍_{B_r(x_0)\cap\Omega} |\beta_{F^*}(x, x_0)|^p \, dx\right)^{\frac{1}{p}} \leq \theta_0 r^\alpha$$

for $x_0 \in \bar{\Omega}$ and $0 < r \ll r_0$.

The second hypothese is that F^* has a $C^{1,1}$ estimate.

Assumption 1.2: We assume that the recession function F^* associated with the operator F exists and has a $C^{1,1}_{\text{loc}}$ estimates. That means $F^*(D^2 v, 0, x) = 0$ in Ω in the viscosity sense, implies $v \in C^{1,1}(\Omega')$ and for $\Omega^{'} \subset \Omega$ we have

$$\|v\|_{C^{1,1}(\Omega')} \leq C\|v\|_{L^\infty(\Omega)}$$

for a constant $C \geq 1$.

Then, we obtain the interior $W^{2,p}$ regularity of the operator F by assuming that the F^* has a better regularity which is $C^{1,1}$ estimate. Our main result states the following:

Theorem 1.3: (A priori $W^{2,p}$ estimate). Let $F: S(d) \times R^d \times \Omega \to R$ satisfy (1.2) and $u \in C(B_1)$ be a viscosity solution of

$$F(D^2 u, Du, x) = f(x) \quad \text{in } B_1 \tag{1.6}$$

where $f \in L^p, p > d$. Assume recession function F^* has Assumption 1.2. Furthermore, suppose that β_{F^*} satisfies Assumption 1.1, i.e.,

$$\left(\frac{1}{|B_r|}\int_{B_r} |\beta_{F^*}(x_0, x)|^n \, dx\right)^{\frac{1}{n}} \leq C r^\alpha$$

for every $x_0 \in B_1$ and some $C > 0$, $\alpha \in (0,1)$.

Then $u \in W^{2,p}(B_{1/2})$ and there exists universal constant $C_0 > 0$ so that

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C_0\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right). \tag{1.7}$$

Remark 1.4: Compared to Caffarelli's results [1], our result accommodates a fairly general class of fully nonlinear operators F. We establish the local $W^{2,p}$ estimate under the weak assumption that the gradient of solution has L^p_{loc} growth rate, which generalizes the results in [14].

The remainder of this paper is organized as follows. Section 2 contains some notions and preliminary results. In Section 3, we use the structural condition and derive the $W^{2,\delta}$ estimates for solutions to S -classes. In Section 4, we obtain $W^{2,p}$ estimates by approximation argument.

2. Preliminaries

In this section, we introduce some definitions and conclusions which will be used in this paper. Firstly, we give the definition of L^p-viscosity solution.

Definition 2.1: Let $F: S(d) \times R^d \times \Omega \backslash N \to R$ be a measured function. Suppose F does not increase with respect to M and let $f \in L^p(\Omega)$, for $p > d/2$. A function $u \in USC(\Omega)$ (That is, upper semi-continuous.) is an L^p-viscosity subsolution to

$$F(D^2u, Du, x) = f(x) \quad in \quad \Omega, \tag{2.1}$$

if, for every $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ such that there exist $\varepsilon > 0$ and an open set $U \subset \Omega$ for which

$$F(D^2\phi(x), D\phi(x), x) - f(x) \geq \varepsilon$$

a.e. $x \in U$, then $u - \phi$ does not attain a local maximum in U. A function $u \in LSC(\Omega)$ (That is, lower semi-continuous.) is an L^p-viscosity supersolution to (2.1) if for every $\phi \in W^{2,p}_{\mathrm{loc}}(\Omega)$ such that there exist $\varepsilon > 0$ and an open set $U \subset \Omega$ for which

$$F(D^2\phi(x), D\phi(x), x) - f(x) \leq -\varepsilon$$

a.e. $x \in U$, then $u - \phi$ does not attain a local minimum in U. If $u \in C(\Omega)$ is simultaneously an L^p -viscosity subsolution and a supersolution to (2.1), we say u is an L^p-viscosity solution to (2.1).

In order to define the set of viscosity solutions of a certain class of uniformly elliptic equations, we introduce the following Puccis extremal operators.

Definition 2.2: Let $0 < \lambda \leq \Lambda$. For any $M \in S$, we define

$$\begin{aligned} M^-(M, \lambda, \Lambda) &= M^-(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i, \\ M^+(M, \lambda, \Lambda) &= M^+(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i, \end{aligned}$$

where $\{e_i\}_{i \geq 1}$ are the eigenvalues of M.

For any $A = (a_{ij}) \in S_{\lambda,\Lambda}$, define a linear functional L_A on S by

$$L_A M = \sum_{i,j=1}^{d} a_{ij} M_{ij} = tr(AM), \quad \forall M \in S.$$

Then it can be seen that

$$\begin{aligned} \mathrm{M}^-(M, \lambda, \Lambda) &= \inf_{A \in \mathrm{S}_{\lambda,\Lambda}} L_A M = \inf_{A \in \mathrm{S}_{\lambda,\Lambda}} tr(AM), \\ \mathrm{M}^+(M, \lambda, \Lambda) &= \sup_{A \in \mathrm{S}_{\lambda,\Lambda}} L_A M = \sup_{A \in \mathrm{S}_{\lambda,\Lambda}} tr(AM). \end{aligned}$$

Some basic properties of Pucci's extremal operators are listed in [17]. Now we introduce S -classes as follows:

Definition 2.3: Let $b, f \in L^d(\Omega)$ with b ≥ 0 and $0 < \lambda \leq \Lambda$ two positive constants. We denote by $\underline{S}(\lambda, \Lambda, b, f)$ the class of functions $u \in C(\Omega)$ such that

$$M^+(D^2u, \lambda, \Lambda) + b(x)|Du| \geq f(x)$$

in the viscosity sense in Ω.

Similarly, $\bar{S}(\lambda, \Lambda, b, f)$ denotes the class of functions $u \in C(\Omega)$ such that

$$M^-(D^2u, \lambda, \Lambda) - b(x)|Du| \le f(x)$$

in the viscosity sense in Ω.

We also define

$$\begin{aligned} S(\lambda, \Lambda, b, f) &= \underline{S}(\lambda, \Lambda, b, f) \cap \bar{S}(\lambda, \Lambda, b, f), \\ S^*(\lambda, \Lambda, b, f) &= \underline{S}(\lambda, \Lambda, b, -|f|) \cap \bar{S}(\lambda, \Lambda, b, |f|). \end{aligned}$$

For simplicity, we will denote $M^-(M, \lambda, \Lambda)$ as $M^-(M)$ and $M^+(M, \lambda, \Lambda)$ as $M^+(M)$ in our paper, that is, omitting λ, Λ, S -classes, similarly.

We also have the analogous definition of touching as follows.

Definition 2.4: Let Ω be a bounded domain and u be a continuous function in Ω. We define, for any open set $H \subset \Omega$ and $M > 0$,

$$\begin{aligned} \underline{G}_M(u, H) = \underline{G}_M(H) &= \{x_0 \in H: \exists P \text{ is concave paraboloid of opening } M \text{ such} \\ &\quad \text{that} P(x_0) = u(x_0) \text{and} P(x) \le u(x) \text{forany} x \in H\}, \\ \underline{A}_M(H) &= H \backslash \underline{G}_M(H) \end{aligned}$$

which are measurable sets. We analogously define the set $\bar{G}_M(H) = \bar{G}_M(u, H)$ of points with tangent paraboloid by the above, i.e.,

$$\begin{aligned} \bar{G}_M(u, H) &= \underline{G}_M(-u, H), \\ \bar{A}_M(H) &= H \backslash \bar{G}_M(H). \end{aligned}$$

Let us finally define

$$\begin{aligned} G_M(H) &= \underline{G}_M(H) \cap \bar{G}_M(H), \\ A_M(H) &= H \backslash G_M(H). \end{aligned}$$

Next, we define a function related to it,

$$\Theta(x) := \Theta\big(u, B_{1/2}\big)(x) = inf\big\{M: x \in G_M\big(B_{1/2}\big)\big\} \in [0, \infty], \quad x \in B_{1/2}.$$

In addition to the above definitions, we will also include the foundational conclusions involved in the process of our proof. We have the following Alexandroff-Bakel'man-Pucci-Krylov-Tso estimates for viscosity solutions. This is adapted from Alexandroff-Bakel'man-Pucci estimate in [17].

Theorem 2.5: Let $u \in \bar{S}(b, f)$ in B_1. Assume $u \ge 0$ on ∂B_1. Then

$$\sup_{B_1}\left(u^-\right) \le C\left(\int_{u=\Gamma(u)} \left|f^+\right|^d\right)^{1/d}$$

where C is a constant depending on λ, Λ and PbP_{L^d}, and $\Gamma(u)$ denotes the convex envelope of $-u^-$ in B_2. The function $-u^-$ takes a value of 0 outside B_1.

The next theorem concerns the stability of L^p-viscosity solutions.

Theorem 2.6: Let $p > d - \varepsilon$. Let $(F_n)_{n \in N}, (f_n)_{n \in N} \subset L^p(\Omega)$ and $(u_n)_{n \in N} \subset C(\Omega)$ be sequences such that:

(i) For every $n \in N$, the operator $F_n: (S(d) \times R^d \times \Omega \backslash N) \to R$ satisfies (1.2).

(ii) For every $n \in N$, the function u_n is an L^p-viscosity solution to

$$F_n(D^2 u_n, Du_n, x) = f_n \quad \text{in} \quad \Omega.$$

Suppose there exists $u_\infty \in C(\Omega)$ such that $u_n \to u_\infty$ locally uniformly as $n \to \infty$. Suppose further there are F_∞ and f_∞ such that, for every $B_r(x_0) \subset \Omega$ and $\varphi \in W^{2,p}(B_r(x_0))$ the function

$$\begin{aligned} g_n(x) := \quad & F_n(D^2\varphi(x), D\varphi(x), x) - F_\infty(D^2\varphi(x), D\varphi(x), x) \\ & + f_\infty(x) - f_n(x) \end{aligned}$$

converges to zero in $L^p(B_r(x_0))$, as $n \to \infty$. Then u_∞ is an L^p-viscosity solution to

$$F_\infty(D^2 u_\infty, Du_\infty, x) = f_\infty \quad \text{in} \quad \Omega.$$

The proof of stability theory of viscosity solutions can be found in [18] and [2]. Next, we obtain the following property by applying the structure condition.

Proposition 2.7: Suppose F satisfies (1.2). Let u be a viscosity subsolution of $F(D^2u, Du, x) = f(x), \phi \in W^{2,p}_{loc}(\Omega)$. Then $u - \phi \in \underline{S}(b, f - F(D^2\phi, D\phi, x))$.

Proof. Starting from the definition, let's proceed with the proof. Let $\psi \in W^{2,p}_{\text{loc}}(\Omega)$ such that $M^+(D^2\psi) + b(x)|D\psi| - f(x) + F(D^2\phi, D\phi, x) < -\varepsilon$ almost everywhere in V for some open set V and $\varepsilon > 0$. By (1.2),

$$\begin{aligned} & F(D^2(\psi + \phi), D(\psi + \phi), x) - f \\ \leq \quad & M^+(D^2\psi) + b(x)|D\psi| + F(D^2\phi, D\phi, x) - f \\ < \quad & -\varepsilon \end{aligned} \tag{2.2}$$

almost everywhere in V. By the definition of viscosity subsolutions, $u - (\psi + \phi)$ takes no local maximum in V, i.e., $(u - \phi) - \psi$ takes no local maximum in V. Therefore $u - \phi \in \underline{S}(b, f - F(D^2\phi, D\phi, x))$.

The following result is a key lemma in our approach. It describes precisely how the point-wise convergence of F_μ to F^* takes place.

Lemma 2.8: (Local uniform convergence) Let F satisfy Assumption 1.2 and assume F^* exists. Then, given $\varepsilon > 0$ there exists $\mu_0 > 0$ such that, for every $\mu < \mu_0$ there holds

$$\frac{|\mu F(\mu^{-1}X, 0, x) - F^*(X, 0, x)|}{1 + \|X\|} \leq \varepsilon \tag{2.3}$$

for every $X \in S(d)$.

Proof. The proof of Lemma 2.8 is elementary, but we carry it out in order to facilitate readers to have a more accurate understanding of the convergence process. Since the function F satisfies (1.2), combining the definition of Pucci's extremal operators, we have that

$$F(X + Y, 0, x) - F(X, 0, x) \leq d\Lambda \|Y\|$$

for elliptic constant Λ, as a result, F is Lipschitz about X. This Lipschitz norm is conserved by the scaling $\mu F(\mu^{-1}X, 0, x)$. By the Arzela-Ascoli theorem we have that up to a subsequence $\mu F(\mu^{-1}X, 0, x)$ converges uniformly in every compact set. Since $\mu F(\mu^{-1}X, 0, x)$ converges pointwise to F^*, then all its subsequential limits must be the same and therefore it converges to F^* uniformly over every compact set. That means that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that

$$\|\mu F(\mu^{-1}X, 0, x) - F^*(X, 0, x)\| \leq \varepsilon,$$

for all matrices X such that $\|X\| \leq 1$ and $\forall \mu < \delta$. This already shows that (2.3) holds if $\|X\| \leq 1$. Now let X be a matrix with $\|X\| > 1$. For any $\mu < \delta$, we can consider also $\mu_1 = \|X\|^{-1}\mu < \mu < \delta$. Therefore

$$\left\|\mu_1 F\left(\mu_1^{-1}\frac{X}{\|X\|}, 0, x\right) - F^*\left(\frac{X}{\|X\|}, 0, x\right)\right\| \leq \varepsilon.$$

Observing that $\mu_1^{-1}\frac{X}{\|X\|} = \mu^{-1}X$, and using that F^* is homogeneous of degree one, we obtain

$$\|\mu F(\mu^{-1}X, 0, x) - F^*(X, 0, x)\| \leq \varepsilon\|X\|.$$

This proves (2.3) for $\|X\| > 1$.

3. $W^{2,\delta}$-Estimate for S -Classes

Our goal in this section is to demonstrate the $W^{2,\delta}$-estimate of viscosity solutions to S -classes. We first study the decay of $|A_t|$ when $u \in S(b, f)$ in B_1 by using barrier function and Calderón-Zygmund decomposition. We will prove that $|A_t|$ has a power decay in t. In particular, we get the following results.

Proposition 3.1: There exists a universal $\delta > 0$ such that if $u \in S(b, f)$ in B_1 and $f \in L^d(B_1)$, then $u \in W^{2,\delta}(B_{1/2})$ and moreover,

$$\|u\|_{W^{2,\delta}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}\right),$$

where C is a universal constant.

To prove Proposition 3.1, we need the following lemmas.

Lemma 3.2: Let g be a nonnegative and measurable function in Ω and $\mu_g$be its distribution function, i.e.,

$$\mu_g(t) = |\{x \in \Omega : g(x) > t\}|, \quad t > 0.$$

Let $\eta > 0$ and $M > 1$ be constants. Then, for $0 < p < \infty$,

$$g \in L^p(\Omega) \Leftrightarrow \sum_{k\geq 1} M^{pk}\, \mu_g(\eta M^k) = S < \infty$$

and

$$C^{-1}S \leq \|g\|_{L^p(\Omega)}^p \leq C(|\Omega| + S),$$

where $C > 0$ is a constant depending only on η, M and p.

For the proof of Lemma 3.2, we refer the reader to [17]. We also introduce two tools that will be useful to prove the decay of $|A_t|$. We first introduce a barrier function.

Lemma 3.3: Given $0 < \lambda \leq \Lambda$ constants, there exists a smooth function ϕ in R^d and universal positive constants C and $M > 1$ such that (recall $\bar{Q}_1 \subset \bar{Q}_3 \subset B_{2\sqrt{d}}$)

$$\begin{aligned} \phi \quad &\geq 0 \quad \text{in} \mathrm{R}^d \backslash B_{2\sqrt{d}}, \\ \phi \quad &\leq -2 \quad \text{in} Q_3, \\ M^+(D^2\phi) + b|D\phi| \quad &\leq C\xi \quad \text{in} \mathrm{R}^d, \end{aligned}$$

where $0 \leq \xi \leq 1$ is a continuous function in R^d with $supp\xi \subset \bar{Q}_1$. Moreover, $\phi \geq -M$ in R^d.

The proof of above lemma can be referred to Lemma 4.1 in [17], and notice that $|D\phi| \leq C$ when $|x| \geq \frac{1}{4}$, here C is a universal constant.

The second tool is a corollary by using the Calderón-Zygmund decomposition. Let Q_1 be the unit cube and Q be a dyadic cube different from Q_1. We say that $\tilde{Q}$ is the predecessor of Q if Q is one of the 2^n cubes obtained from dividing $\tilde{Q}$.

Lemma 3.4: Let $A \subset B \subset Q_1$ be measurable sets and $0 < \delta < 1$ such that

(1) $|A| \leq \delta$,

(2) if Q is a dyadic cube such that $|A \cap Q| > \delta|Q|$, then $\tilde{Q} \subset B$.

Then $|A| \leq \delta|B|$.

The proof of this lemma is referenced in [17], [19]. Now we provide an estimate of the lower bound on the measure of the contact set for u.

Lemma 3.5: Assume that Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Let u be continuous in Ω satisfying $\|u\|_{L^\infty(\Omega)} \leq 1$, $u \in \bar{S}(b, f)$ in $B_{6\sqrt{d}}$ and $\|f\|_{L^d\left(B_{6\sqrt{d}}\right)} \leq \delta_0$. Then

$$\left|\underline{G}_M(u, \Omega) \cap Q_1\right| \geq 1 - \sigma, \tag{3.1}$$

where $0 < \sigma < 1$, $\delta_0 > 0$ and $M > 1$ are universal constants.

Proof. Consider the barrier function ϕ of Lemma 3.3. Let $w = u + 2\phi + 1$ in $\bar{B}_{2\sqrt{d}}$. Since $\phi \geq 0 \text{in} \mathrm{R}^d \backslash B_{2\sqrt{d}}$, $\phi \leq -2 \text{in} Q_3$, clearly $w \geq 0$ on $\partial B_{2\sqrt{d}}$ and $\inf_{Q_3} w \leq -2$. Applying the ABP estimate to w we get

$$|\{w = \Gamma_w\} \cap Q_1| \geq 1 - \sigma. \tag{3.2}$$

Since $u \in \bar{S}(b, f)$, ϕ satisfies $M^+(D^2\phi) + b|D\phi| \leq C\xi$, then $u + 2\phi + 1 \in \bar{S}(b, |f| + C\xi)$, by [20] (Proposition 3.7). So we can apply ABP estimate to w again, i.e.,

$$1 \leq C(\int_{\{w=\Gamma_w\} \cap B_{2\sqrt{d}}} (|f| + C\xi)^d)^{1/d}.$$

It remains to prove that $(\{w = \Gamma_w\} \cap Q_1) \subset (\underline{G}_M(\Omega) \cap Q_1)$, for some $M > 1$ universal.

To see this, let $x^* \in \{w = \Gamma_w\} \cap Q_1$. It follows that there exists an affine function L such that $L < 0$ on $\partial B_{2\sqrt{d}}$ (Γ_w is the convex envelope with respect to $B_{4\sqrt{d}}$, and hence $\Gamma_w < 0$ in $B_{4\sqrt{d}}$) and

$$L \leq \Gamma_w \leq w = u + 2\phi + 1 \text{in} B_{2\sqrt{d}},$$
$$L(x^*) = \Gamma_w(x^*) \leq w(x^*).$$

Since the C^2 norm of ϕ in $\bar{B}_{2\sqrt{d}}$ is bounded by a universal constant, it follows that there is a concave paraboloid P of opening M such that

$$P \leq L - 2\phi - 1 \leq u \text{in} B_{2\sqrt{d}}, \text{ with equalities at } x^*. \tag{3.3}$$

Since $L < 0$ and $\phi \geq 0$ on $\partial B_{2\sqrt{d}}$, we have that $P < -1$ on $\partial B_{2\sqrt{d}}$. Since $P(x^*) = u(x_0) \geq -1, x^* \in B_{2\sqrt{d}}$ and $\{x \in R^d : P(x) \geq -1\}$ is convex, it follows that $P < -1$ in $R^d \backslash B_{2\sqrt{d}}$.

Therefore $P \leq u$ in $\Omega \backslash B_{2\sqrt{d}}$. This combined with (3.3), imply that $x_0 \in \underline{G}_M(\Omega) \cap Q_1$.

Furthermore, we can use Lemma 3.5 to obtain the following lemma, whose proof is similar to Lemma 7.6 in [17].

Lemma 3.6: Assume that Ω is a bounded domain such that $B_{6\sqrt{d}} \subset \Omega$. Let u be continuous in Ω satisfying $u \in \bar{S}(b, f)$ in $B_{6\sqrt{d}}$, $\|f\|_{L^d\left(B_{6\sqrt{d}}\right)} \leq \delta_0$ and $\underline{G}_1(u, \Omega) \cap Q_3 \neq \emptyset$. Then

$$\left|\underline{G}_M(u,\Omega)\cap Q_1\right|\geq 1-\sigma,$$

where $0<\sigma<1$, $\delta_0>0$ and $M>1$ are universal constants.

By means of Lemmas 3.4 and 3.6, we get the following lemma whose detailed proof can be referred to [17].

Lemma 3.7: Under the hypotheses of Lemma 3.5, extend f by zero outside $B_{6\sqrt{d}}$ and for k=0,1,2, ..., let

$$\begin{aligned} A &= \underline{A}_{M^{\lambda+1}}(u,\Omega)\cap Q_1, \\ B &= \left(\underline{A}_{M^{\lambda}}(u,\Omega)\cap Q_1\right)\cup\{x\in Q_1: m(f^n)(x)\geq (c_1M^k)^d\}. \end{aligned}$$

Then $|A|\leq\sigma|B|$ where $0<\sigma<1$, δ_0, $M>1$ and c_1 are positive universal constants. Recall that $m(f^d)$ denotes the maximal function of f^d.

With the aim of these lemmas, we prove a result on the decay rate of the measure of certain sets.

Lemma 3.8: Under the hypotheses of Lemma 3.5,

$$\left|\underline{A}_t(u,\Omega)\cap Q_1\right|\leq c_2t^{-\mu},\quad \forall t>0$$

where c_2 and μ are positive universal constants. If, in addition, $u\in S(b,f)$ in $B_{6\sqrt{d}}$, then

$$|A_t(u,\Omega)\cap Q_1|\leq c_2t^{-\mu},\quad \forall t>0.$$

Proof. Define

$$\alpha_k=\left|\underline{A}_{M^k}(\Omega)\cap Q_1\right|,\beta_k=|\{x\in Q_1: m(f^n)(x)\geq (c_1M^k)^n\}.$$

According to Lemma 3.7,

$$\alpha_k\leq\sigma^k+\sum_{i=0}^{k-1}\sigma^{k-i}\beta_i.$$

Since $\|f^n\|_{L^1}\leq\delta_0^n$ and the maximal operator is of weak (1,1) type, we have that

$$\beta_k\leq c(n)\delta_0^n(c_1M^k)^{-n}=CM^{-nk}.$$

Hence

$$\sum_{i=0}^{k-1}\sigma^{k-i}\beta_i\leq C\sum_{i=0}^{k-1}\sigma^{k-i}M^{-ni}\leq Ckm_0^k,\quad m_0=max\{\sigma,M^{-n}\}<1.$$

We conclude that

$$\alpha_k\leq\sigma^k+Ckm_0^k\leq(1+Ck)m_0^k.$$

We conclude this section by combining the above lemmas and obtain the proof of the main result in this section.

Proof of Proposition 3.1. Now we can prove the main conclusion of the third part.

By a standard covering argument, we may assume that $u\in S(b,f)$ in $B_{6\sqrt{d}}$, may as well assume that the solution is formalized, in other words, $\|u\|_{L^\infty\left(B_{6\sqrt{d}}\right)}\leq 1$ and $\|f\|_{L^d\left(B_{6\sqrt{d}}\right)}\leq\delta_0$.

We need to prove $\|D^2u\|_{L^\delta(B_{1/2})}\leq C$. We will take $\delta=\mu/2$. By Lemma 3.8, letting $\Omega=B_{6\sqrt{d}}$, we have that

$$\sum_{k\geq 1}M^{\frac{\mu}{2}k}\left|A_{M^k}\left(B_{6\sqrt{d}}\right)\cap Q_1\right|\leq C,$$

where C is a universal constant. Noticing that $B_{1/2} \subset Q_1 \subset B_{6\sqrt{d}}$, we have that

$$A_{M^k}(B_{1/2}) \subset A_{M^k}(B_{6\sqrt{d}}) \cap Q_1.$$

We conclude that

$$\sum_{k\geq 1} M^{\frac{\mu}{2}\lambda} \left|A_{M^k}(B_{1/2})\right| \leq C.$$

By using Lemma 3.2, we complete the proposition.

4. $W^{2,p}$ Estimates for (1.1)

In this section, we use a geometric tangential path from the original operator F which satisfies (1.2) to the recession operator F^* that has priori $C^{1,1}$ estimates, which could transmit the regularity. What we expect is that such a transfer mechanism has no loss in regularity, but this is not achievable. Thus, without changing the conditions, we can obtain a slightly weaker regularity for original operator F. Firstly, we obtain the approximation lemma though imposing a condition on the behavior of F at the ends of $S(d)$.

Lemma 4.1: (Approximation Lemma) Let $u \in C(B_1)$ be a viscosity solution of

$$F_\mu(D^2u, x) = f(x) \quad \text{in} B_1$$

where F satisfies (1.2), $f \in C(B_1) \cap L^p(B_1)$, $p > d$. Assume recession function F^* satisfies Assumption 1.2.

Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, d, F)$ such that if

$$\|f\|_{L^p(B_1)} \leq \delta \quad \text{and} \quad \mu < \delta,$$

there exists $h \in C^{1,1}(B_{3/4})$ which is a solution of $F^*(D^2h, x) = 0 \text{in} B_{3/4}$, and for some $C > 0$, h satisfies $\|h\|_{C^{1,1}(B_{1/2})} \leq C\|h\|_{L^\infty(b_1)}$ and

$$\|u - h\|_{L^\infty(B_{1/2})} \leq \varepsilon. \tag{4.1}$$

Proof. We prove this lemma by the way of contradiction. Firstly, we suppose the statement does not hold, which means that there exists a sequence of numbers $(\mu_n)_{n\in N}$ and a sequence of functions $(u_n)_{n\in N}$, $(f_n)_{n\in N}$ satisfing that u_n is a viscosity solution of $F_{\mu_n}(D^2u_n, x) = f_n(x)$ in B_1, where $\mu_n \to 0$ and $\|f_n\|_{L^p(B_1)} \to 0$ as $n \to \infty$. However, for every $h \in C^{1,1}(B_1)$ solving $F^*(D^2h, x) = 0$ in B_1, there exists $\varepsilon_0 > 0$ such that $\|u - h\|_{L^\infty(B_{1/2})} \geq \varepsilon_0$.

From [21] we could obtain the standard property which is the compactness about solutions of $F_{\mu_n} = f_n$. Because of the regularity result, we have that $(u_n)_{n\in N} \subset C^\alpha$.

Furthermore, $(u_n)_{n\in N}$ is uniformly bounded in $C^\alpha(B_1)$ and then we apply Arzela-Ascoli theorem for $(u_n)_{n\in N}$. Since $u_n \in C^\alpha$, for some $\alpha \in (0,1)$, we have that $|u_n(x) - u_n(y)| \leq C|x - y|^\alpha$. For every $\varepsilon > 0$, there exists $\theta > 0$ such that $(u_n)_{n\in N} < C\theta^\alpha < \varepsilon$ when $|x - y| < \theta$. So $(u_n)_{n\in N}$ is uniformly equicontinuous. It means that, through a subsequence if necessary, for some $\beta \in (0,1)$ and $\beta < \alpha$, there exists $u_\infty \in C^\beta(B_{3/4})$ such that

$$u_n \to u_\infty \quad \text{inthe} C^\beta_{loc}(B_1) - \text{topology}.$$

On the other hand, from Lemma 2.8, F_{μ_n} converges uniformly in compact sets of $S(d)$ to F^*. The stability of viscosity solutions implies that u_∞ is the solution of $F^*(D^2u_\infty, x) = 0$ in $B_{3/4}$ through $\|f_n\|_{L^p(B_1)} \to 0$ according to Theorem 2.6. Then $u_\infty \in C^{1,1}$ and since $\|u_n - u_\infty\|_{L^\infty(B_{3/4})} \to 0$, as $n \to 0$, we could take $h \equiv u_\infty$. Finally, we get a contradiction and complete the proof.

Next, we produce a lower bound of $|G_M(u, B_1) \cap Q|$, for the universal $M > 1$.

Lemma 4.2: Let $u \in C(B_1)$ be a viscosity solution of $F(D^2u, x) = f$ in B_1 and suppose

$$-|x|^2 \le u(x) \le |x|^2 \quad \text{in} \quad B_1 \backslash B_{3/4}.$$

Under the assumptions of Lemma 4.1, then for some $\rho \in (0,1)$, there exists $M = M(\rho, n) > 0$, such that

$$|G_M(u, B_1) \cap Q_1| \ge 1 - \rho.$$

Proof. According to Lemma 4.1, we can get the function h that is restricted in $B_{1/2}$. Then we extend h outside of $B_{1/2}$ continuously, which could satisfy

$$h = u \quad \text{in} B_1 \backslash B_{3/4} \quad \text{and} \quad \|u - h\|_{L^\infty(B_1)} = \|u - h\|_{L^\infty(B_{3/4})}.$$

Furthermore, since the general maximal principle, for the Dirichlet problem

$$\begin{cases} F^*(D^2h, x) = 0 & \text{in} B_{3/4} \\ \quad h = u & \text{on} \, \partial B_{3/4} \end{cases}$$

we have $\|h\|_{L^\infty(B_{3/4})} \le \|u\|_{L^\infty(\partial B_{3/4})}$.

Then

$$\|u - h\|_{L^\infty(B_1)} = \|u - h\|_{L^\infty(B_{3/4})} \le \|u\|_{L^\infty(B_{3/4})} + \|u\|_{L^\infty(\partial B_{3/4})} \le 2.$$

Therefore,

$$\begin{cases} -|x|^2 \le h(x) \le |x|^2 & \text{in} B_1 \backslash B_{3/4} \\ -2 \le h(x) \le 2 & \text{in} B_{3/4} \backslash B_{1/2} \end{cases}$$

So, we have $-2 - |x|^2 \le h(x) \le 2 + |x|^2 \quad \text{in} B_1 \backslash B_{1/2}$ and from standard results in elliptic regularity theory, we also have $h \in C^2(\bar{B}_{1/2})$.

Because of $Q_1 \subset B_1$ and analysis about h, it's easy to see that h has the paraboloid of bounded opening N touching it for every $x \in B_1$. Then for some $N > 0$, there exists

$$Q_1 \subset G_N(h, B_1). \tag{4.2}$$

Now let $w = \rho_0(u - h)$, where $\rho_0 > 0$. We first apply Proposition 2.7 to u, which could obtain $u - h \in S(\lambda/n, \Lambda, b, f - F(D^2h, Dh, x))$. It also means that

$$w \in S(\lambda/n, \Lambda, \rho_0 b, \rho_0(f - F(D^2h, Dh, x))).$$

So we can use Proposition 3.1 for w and get the following estimate

$$|A_t(w, B_1) \cap Q_1| \le Ct^{-\mu}, \quad \forall t > 0.$$

Then

$$|A_s(u - h, B_1) \cap Q_1| \le C\rho_0^{-\mu} s^{-\mu}$$

when we suppose that $s = t/\rho_0$. Next, we could fix s as N, and for some $\rho > 0$, we have

$$|G_N(u - h, B_1) \cap Q_1| \ge 1 - \rho_0^{-\mu} N^{-\mu} \ge 1 - \rho.$$

Since (4.2),

$$(G_N(u - h, B_1) \cap Q_1) \subset (G_{2N}(u, B_1) \cap Q_1).$$

Then $|G_M(u,B_1)| = |G_{2N}(u,B_1)| \geq 1-\rho$ when we take $M = 2N > 0$.

Under the assumptions of Lemma 4.1 and the relaxation of Lemma 4.2, we can further obtain the lower bound of $|G_M(u,B_1) \cap Q|$ with the same universal constants as follows:

Lemma 4.3: Let $u \in C(B_1)$ be a viscosity solution of $F(D^2u,x) = f$ in B_1 and suppose the assumptions of Lemma 4.1 are in force and

$$G_1(u,B_1) \cap Q_3 \neq \emptyset.$$

Then

$$|G_M(u,B_1) \cap Q_1| \geq 1-\rho,$$

where $M > 0$ and $\rho > 0$ are as same as those in Lemma 4.2.

Proof. Because of $G_1(u,B_1) \cap Q_3 \neq \emptyset$, we could take $x_1 \in G_1(u,B_1) \cap Q_3$. Then there exist affine functions L_1 and L_2 such that

$$\begin{cases} u(x) \leq \dfrac{|x-x_1|^2}{2} + L_1(x) \\ \quad u(x_1) = L_1(x_1) \end{cases} \quad \text{and} \quad \begin{cases} u(x) \geq -\dfrac{|x-x_1|^2}{2} + L_2(x) \\ \quad u(x_1) = L_2(x_1) \end{cases}$$

So we choose the new affine function $L(x)$, which satisfies

$$L_1 \leq L \leq L_2 \quad \text{in} B_1.$$

Then

$$-\frac{|x-x_1|^2}{2} \leq u(x) - L(x) \leq \frac{|x-x_1|^2}{2}. \tag{4.3}$$

Then we suppose that $v(x) = (u(x) - L(x))/c(d)$, where $c(d)$ is a constant depending on the dimension, and we could directly calculate that v is the viscosity solution of

$$G(D^2v,x) = \frac{1}{c(d)} F(c(d)D^2v,x) = \tilde{f} \quad \text{in} B_1$$

where $\tilde{f} := \frac{1}{c(d)} f(x)$.

It's easy to see that new operator G has the same structure condition with F. What's more, $\tilde{f}$ is L^p since $f \in L^p(B_1)$, $p > d$. According to (4.3), we have

$$-\frac{|x-x_1|^2}{2c(d)} \leq v(x) \leq \frac{|x-x_1|^2}{2c(d)}.$$

Let $c(d)$ be enough large, such that $\|v\|_{L^\infty(B_1)} \leq 3/4$ and $-|x|^2 \leq v(x) \leq |x|^2$ in $B_1 \backslash B_{3/4}$. By applying Lemma 4.2, we obtain $|G_M(v,B_1) \cap Q_1| \geq 1-\rho$, where ρ and M are same with Lemma 4.2. Then

$$|G_{c(d)M}(u,B_1) \cap Q_1| \geq 1-\rho.$$

So we update the parameter M for Lemma 4.2 by taking $M = max\{M, c(d)M\}$.

Now, we ready to prove the following lemma which is a key ingredient in our approach.

Lemma 4.4: Let $u \in C(B_1)$ be a viscosity solution of $F_\mu(D^2u,x) = f$ in B_1 and suppose the assumptions of Lemma 4.1 are in force. Extend f outside of B_1 by zero. Define

$$A := A_{M^{k+1}}(u, B_1) \cap Q_1$$

and

$$B := (A_{M^k}(u, B_1) \cap Q_1) \cup \{x \in Q_1 | m(f^d)(x) \geq (cM^k)^d\}.$$

Suppose further there exists $\delta > 0$, yet to be determined, such that $\mu + \|f\|_{L^d(B_1)} \leq \delta$. Then there exists $\sigma \in (0,1)$ such that

$$|A| \leq \sigma |B|.$$

Proof. We will prove that sets A and B satisfy two conditions of Lemma 3.4. Firstly, it's easy to find that $\|u\| \leq 3/4 \leq |x|^2$ in $B_1 \backslash B_{3/4}$. Therefore, by recurring to Lemma 4.2, we have $|G_M(u, B_1) \cap Q_1| \geq 1 - \rho$ and since $(G_M(u, B_1) \cap Q_1) \subset \left(G_{M^{k+1}}(u, B_1) \cap Q_1\right)$, then

$$|G_{M^{k+1}}(u, B_1) \cap Q_1| \geq 1 - \rho.$$

So we could verify the first term $|A| \leq \rho$. Next, we will check the following claim, which is that if Q is a dyadic cube of Q_1, and satisfies $|A \cap Q| \geq \rho |Q|$, then $\tilde{Q} \subset B$, where $\tilde{Q}$ is the parent generation of Q. We could suppose $Q = Q_{1/2^i}(x_0)$ for some $i \geq 0$ and $x_0 \in Q_1$. Notice that

$$|A_{M^{k+1}}(u, B_1) \cap Q| \geq |A \cap Q| \geq \rho |Q|. \tag{4.4}$$

We prove this claim by the way of contradiction. Suppose $\tilde{Q}$ is not subset of B. It also means that there exists $x_1 \in \tilde{Q}$ and $x_1 \notin B$, then

$$x_1 \in \tilde{Q} \cap G_{M^k}(u, B_1) \quad \text{and} \quad m(f^d)(x_1) \leq (CM^k)^d. \tag{4.5}$$

We define $\tilde{u}$ as

$$\tilde{u}(y) := \frac{2^{2i}}{M^k} u(x_0 + \frac{1}{2^i} y),$$

and for any $y \in B_1$, we have $x_0 + \frac{1}{2^i} y \in B_{1/2^i}(x_0)$. Through directly calculating, we know that $\tilde{u}$ is the viscosity solution of

$$G(D^2 \tilde{u}, x) = \tilde{f},$$

where

$$G(D^2 u) = \frac{1}{M^k} F(M^k D^2 u) \quad \text{and} \quad \tilde{f}(y) = \frac{1}{M^k} f(x_0 + \frac{1}{2^i} y).$$

Since $x_1 \in \tilde{Q} = Q_{1/2^{i-1}}(x_0)$,

$$|x_1 - x_0|_\infty \leq \frac{3}{2^{i+1}}.$$

So

$$B_{1/2^i}(x_0) \subset Q_{5/2^i}(x_1).$$

Then, by using the variable substitution and (4.5), we have

$$\|\widetilde{f}\|^d_{L^d(B_1)} \leq \frac{2^{2i}}{M^{kd}} \int_{Q_{5/2^i}(x_1)} |f(x)|^d \, dx \leq 2^{(2-d)i} 5^d C^d \leq C_d C^d.$$

Then we take C enough small, such that $C_d C^d \le \rho^d$ and since (4.5),

$$G_1(\tilde{u}, B_{1/2^i}(x_0)) \cap Q_3 \neq \emptyset.$$

So we could apply Lemma 4.3 for $\tilde{u}$, one obtains

$$|G_M(\tilde{u}, B_{1/2^i}(x_0)) \cap Q_1| \ge 1-\rho = (1-\rho)|Q_1|.$$

Then

$$|G_{M^{k+1}}(u, B_1) \cap Q| \ge (1-\rho)|Q|,$$

it is in contradiction with (4.4).

Then, we apply the approximation method to prove the local $W^{2,p}$ estimate for continuity and integrability of the source term respectively.

Proposition 4.5: Let $u \in C(B_1)$ be a viscosity solution of

$$F(D^2u, x) = f(x) \quad \text{in} B_1$$

where $f \in C(B_1) \cap L^p(B_1)$, $p > d$. Assume recession function F^* satisfies Assumption 1.2. Furthermore, suppose that β_{F^*} satisfies $\|\beta_{F^*}(x_0, \cdot)\|_{L^p(B_1)} = 1/2$, for every $x_0 \in B_1$.

Then $u \in W^{2,p}(B_{1/2})$ and there exists universal constant $C > 0$ so that

$$\|u\|_{W^{2,p}(B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right).$$

Proof. For $k \ge 0$, set

$$\alpha_k := |A_{M^k}(u, B_1) \cap Q_1|$$

and

$$\beta_k := |\{x \in Q_1 | m(f^d)(x) \ge (CM^k)^d\}|.$$

According to Lemma 4.4, we have $\alpha_{k+1} \le \rho(\alpha_k + \beta_k)$ and through the recursion calculation,

$$\alpha_k \le \rho^k + \sum_{i=0}^{k-1} \rho^{k-i}\beta_i. \tag{4.6}$$

Moreover, since $f^d \in L^{p/d}$,

$$m(f^d) \in L^{p/d} \quad \text{and} \quad \|m(f^d)\|_{L^{p/d}} \le C\|f\|_{L^d}^d \le C.$$

Then, by Lemma 3.2,

$$\sum_{k\ge 0} M^{pk}\,\beta_k \le C. \tag{4.7}$$

Because of $B_{1/2} \subset Q_1$, one obtains $\mu_\theta(t) \le |A_t(B_{1/2})| \le |A_t(B_{1/2}) \cap Q_1|$, so we just need to prove

$$\sum_{k\ge 1} M^{pk}\,\alpha_k \le C \tag{4.8}$$

by recurring once again to Lemma 3.2. Now we determine ρ by taking $\rho M^p = 1/2$, and by (4.6) and (4.7), then we have

$$\begin{aligned}\sum_{k\ge1} M^{pk}\,\alpha_k &\le \sum_{k\ge1} M^{pk}\left(\rho^k+\sum_{i=0}^{k-1}\rho^{k-i}\beta_i\right)\\ &\le \sum_{k\ge1}(\rho M^p)^k+\sum_{k\ge1}\sum_{i=0}^{k-1}\rho^{k-i}\,M^{p(k-i)}M^{pi}\beta_i\\ &=\sum_{k\ge1}2^{-k}+\left(\sum_{i\ge0}M^{pi}\,\beta_i\right)\left(\sum_{j\ge1}2^{-j}\right)\\ &=\left(\sum_{k\ge1}2^{-k}\right)\left(1+\sum_{k\ge0}M^{pk}\,\beta_k\right)\\ &\le C.\end{aligned}$$

Finally, by using approximation methods for existing $W^{2,p}$ estimate about continuous source terms, it is possible to obtain that about the integrable source term.

Proof of Theorem 1.3. Suppose u is the viscosity solution of (1.6). Define $g(x)=F(D^2u,0,x)$ for every $x\in B_1$. Since [22] (Proposition 3.2), we obtain that u is parabolical twice differentiable a.e. and its pointwise derivatives satisfy (1.6) a.e. in B_1. According to the structure condition and trigonometric inequality, we have

$$\begin{aligned}|g(x)| &\le |F(D^2u,0,x)-F(D^2u,Du,x)|+|f(x)|\\ &\le b(x)|Du|+|f(x)|.\end{aligned}$$

Therefore, former results on the regularity of continuous viscosity solutions imply $g\in L^p(B_1)$, since [22] (Theorem 7.3). Set

$$G(D^2u,x):=F(D^2u,0,x).$$

By using Proposition 4.1 of [22], one obtains that u is an L^p-viscosity solution of $G(D^2u,x)=g(x)$ in B_1. Then we could apply Proposition 4.5 to u and get the $W^{2,p}$ estimate of the viscosity solution of

$$F(D^2u,Du,x)=f(x)\quad \text{in} B_1$$

where $f\in C(B_1)\cap L^p(B_1)$.

Now, we consider the problem about $f\in L^p(B_1)$. So, the continuity of f is unknown on this condition. We adopt the approximate method. Thus, we think about a sequence of functions $(g_n)_{n\in N}\subset C(B_1)\cap L^p(B_1)$ satisfying

$$\|g_n-g\|_{L^p(B_1)}\to 0\quad \text{as}\quad n\to\infty, \tag{4.9}$$

and a sequence of functions $(u_n)_{n\in N}$ satisfying that u_n is a viscosity solution of the following family of equations

$$G(D^2u_n,x)=g_n(x)\quad \text{in} B_1.$$

Though simple verification, G^* has a priori $C^{1,1}$ estimates and satisfies $\|\beta_{G^*}(x_0,\cdot)\|_{L^p(B_1)}=1/2$, for every $x_0\in B_1$. Then, we have

$$\|u_n\|_{W^{2,p}_{loc}(B_1)}\le C(\|u_n\|_{L^\infty(B_1)}+\|g_n\|_{L^p(B_1)}). \tag{4.10}$$

There exists $\bar u\in C(\bar B_1)$ such that $u_n\to u$ in $C(\bar B_1)$ though using Proposition 2.6 of [22]. We also find that u_n weakly converges to $\bar u$ in $W^{2,p}_{loc}(B_1)$. Hence, by using the stability of viscosity solutions, $\bar u$ is an L^p-viscosity solution of $G(D^2u,x)=g(x)$ in B_1 and from (4.9) and (4.10), we have

$$\|\bar u\|_{W^{2,p}_{loc}(B_1)}\le C(\|\bar u\|_{L^\infty(B_1)}+\|g\|_{L^p(B_1)}).$$

Finally, we apply the maximum principle which yields $\bar u=u$. We complete the proof of our theorem.

Conflict of Interest

The authors declare no conflict of interest.

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