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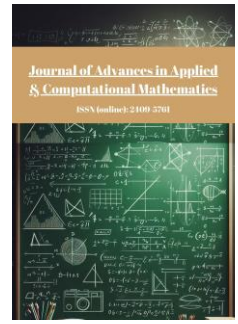

Fractional Inequalities for Exponentially s-Convex Functions on Time Scales

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ABSTRACT

In this paper, we present new integral inequalities involving exponentially s-convex functions in the second sense on time scales. By utilizing the delta Riemann-Liouville fractional integral and the fractional Taylor formula, we establish upper bounds for functions that are n-times rd-continuously Δ-differentiable with exponentially s-convex properties. Our results provide novel insights into the theory of time scales, bridging the gap between discrete and continuous calculus. The application of fractional calculus on time scales is explored, and several well-known inequalities are employed to derive the main findings. These results have potential implications for further studies in fractional dynamic calculus and other related fields.

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1. Introduction

The theory of time scale is a relatively new branch of mathematics which was initiated in 1988, the German mathematician Stefan Hilger [1] proposed a time scale as a unifier between the discrete and continuous calculus. Since then, it has gained a lot of interest from mathematicians working in various fields of the mathematical sciences, which among those for instance we can refer to [2-5] that are devoted to develop various results concerning fractional calculus on time scales.

In the present article, motivated by the methods in [6-9], we attempt to prove some upper bounds for the delta-Riemann-Liouville fractional integral of functions which are n -times rd -continuously Δ -differentiable with exponentially s -convexity property in the second sense, on an interval in some time scales. In the next section, we give some basic results and well-known inequalities which are useful in proving our main results. In Section 3, the main results are framed and justified anchored on the referred results, especially the notion of exponentially s -convexity in the second sense, fractional Taylor formula and the technical lemma which have the main role among the others.

2. Preliminaries

Suppose that T is an unbounded time scale with forward jump operator and delta differentiation operator σ and Δ -respectively. Let also, $a, b \in T$, $a < b$ and an interval $[a, b]$ in T means as an intersection of a real interval with the supposed time scale.

For $\alpha \geq 0$, with $h_\alpha: T \times T \to R$ we will denote the generalized polynomials on time scales defined as follows

$$h_0(t, s) = 1,$$

$$h_\alpha(t, s) = \int_s^t h_{\alpha-1}(t, \sigma(\tau))\Delta\tau, \quad t, s \in \mathrm{T}.$$

Furthermore, it is established in [7] that for $\alpha, \beta \geq 1$ we have

$$\int_a^t h_{\alpha-1}(t, \sigma(u))h_{\beta-1}(u, a)\Delta u = h_{\alpha+\beta-1}(t, a), \quad t \in [a, b]. \tag{2.1}$$

For $\alpha \geq 1$ and $f \in C_{rd}(T)$, with D_a^α we denote the delta-Riemann-Liouville fractional operator defined by

$$D_a^\alpha f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau))f(\tau)\Delta\tau,$$

$$D_a^0 f(t) = f(t), \quad t \in \mathrm{T}.$$

We note that $C_{rd}(T)$ is the set of all rd -continuous functions $f: T \to R$.

We will start with the following useful auxiliary results.

Lemma 2.1 [11] Let $\alpha, \beta > 1$, $f \in \mathrm{C}_{rd}([a, b])$. Then

$$D_a^\alpha D_a^\beta f(t) = D_a^{\alpha+\beta} f(t)$$

$$+ \int_a^t f(u)\mu(u)h_{\alpha-1}(t, \sigma(u))h_{\beta-1}(u, \sigma(u))\Delta u,$$

where $t \in [a, b]$ and μ is the graininess function, i.e., $\mu(t) = \sigma(t) - t$.

Definition 2.2 [11] Let $\alpha, \beta > 1$, $f \in \mathrm{C}_{rd}([a, b])$. The integral

$$E(f, \alpha, \beta, t) = \int_a^t f(u)\mu(u)h_{\alpha-1}(t, \sigma(u))h_{\beta-1}(u, \sigma(u))\Delta u, \quad t \in [a, b],$$

is called the forward graininess deviation functional of f.

By Lemma 2.1, we have

$$D_a^\alpha D_a^\beta f(t) = D_a^{\alpha+\beta} f(t) + E(f,\alpha,\beta,t), \quad t \in [a,b].$$

Definition 2.3 [10] Let $\alpha > 2$ and $m-1 < \alpha < m$, $m \in \mathrm{N}$, $\nu = m - \alpha$. For a function $f \in \mathrm{C}_{rd}^m([a,b])$, define

$$\Delta_a^{\alpha-1} f(t) = D_a^{\nu+1} f^{\Delta^m}(t)$$

$$= \int_a^t h_\nu(t,\sigma(u)) f^{\Delta^m}(u)\Delta u, \quad t \in [a,b].$$

Lemma 2.4 [10] Suppose all as in Definition 2.3. Then

$$\int_a^t h_{m-1}(t,\sigma(\tau)) f^{\Delta^m}(\tau)\Delta\tau = -\int_a^t f^{\Delta^m}(u)\mu(u)h_{\alpha-2}(t,\sigma(u))h_\nu(u,\sigma(u))\Delta u$$

$$+ \int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta_a^{m-1} f(\tau)\Delta\tau, \quad t \in [a,b].$$

According to the Taylor formula on time scales and Lemma 2.4, we have the following identity.

Lemma 2.5 (Fractional Taylor Formula) [10] Under the conditions of Definition 2.3, we have

$$f(t) = \sum_{k=0}^{m-1} h_k(t,a) f^{\Delta^k}(a)$$

$$- \int_a^t f^{\Delta^m}(u)\mu(u)h_{\alpha-2}(t,\sigma(u))h_\nu(u,\sigma(u))\Delta u$$

$$+ \int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta_a^{\alpha-1} f(\tau)\Delta\tau, \quad t \in [a,b].$$

Definition 2.6 [10] For the same assumptions as above, i.e, for $\alpha > 2$ and $m-1 < \alpha < m$, $m \in \mathrm{N}$, $\nu = m - \alpha$, and for a function $f \in \mathrm{C}_{rd}^m([a,b])$, define

$$B(t) = f(t) + E\left(f^{\Delta^m}, \alpha-1, \nu+1, t\right), \quad t \in [a,b].$$

By the fractional Taylor formula the following result is established.

Lemma 2.7 [11] Let all as in Definition 2.5. In addition, let $f^{\Delta^k}(a) = 0$, $k \in \{0,\dots,m-1\}$. Then

$$B(t) = \int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta_a^{\alpha-1} f(\tau)\Delta\tau, \quad t \in [a,b].$$

Applying the fractional Taylor formula we has proved the following equality earlier, which has the main role in deriving our results. For the readers' convenience we bring the proof.

Lemma 2.8 [9] Let $f \in \mathrm{C}_{rd}^{m-1}([a,b])$, $\alpha > 2$, $m-1 < \alpha < m$, $\nu = m - \alpha$. Then

$$\int_a^t f(s)\Delta s = \sum_{k=0}^{m-1} h_k(t,a) f^{\Delta^k}(a)$$

$$- \int_a^t f^{\Delta^{m-1}}(u)\mu(u)h_{\alpha-2}(t,\sigma(u))h_\nu(u,\sigma(u))\Delta u$$

$$+ \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau h_\nu(\tau,\sigma(u)) f^{\Delta^{m-1}}(u)\Delta u\right)\Delta\tau,$$

$t \in [a,b]$.

Proof. Let

$$g(t) = \int_a^t f(s)\Delta s, \quad t \in [a,b].$$

Then

$$g^{\Delta^k}(t) = f^{\Delta^{k-1}}(t), \quad k \in \{1, \dots, m\},$$

$$\Delta_a^{\alpha-1} g(t) = \int_a^t h_\nu(t, \sigma(u)) f^{\Delta^{m-1}}(u)\Delta u, \quad t \in [a,b].$$

We apply the fractional Taylor formula for the function g and we get the desired result. This completes the proof.

The following newly concept was introduced in [8].

Definition 2.9 Let $s \in (0,1]$, $x_0 \in [a,b]$. A function $f: [a,b] \to \mathrm{R}$ is called exponentially s -convex in the second sense if

$$f(t) \leq \left(\frac{b-t}{b-a}\right)^s \frac{f(a)}{e_\beta(a,x_0)} + \left(\frac{t-a}{b-a}\right)^s \frac{f(b)}{e_\beta(b,x_0)} \tag{2.2}$$

for any $t \in [a,b]$ and for some $\beta \in R_+$. If (3.1) holds in the reverse sense, then we say that f is exponentially s -concave in the second sense.

Throughout this paper, without loss of generality, suppose that $s \in (0,1)$.

We need the following technical lemma.

Lemma 2.10 [8] *We have*

$$\int_a^b (b-t)^s \Delta t \leq (b-a)^{1-s}(h_2(a,b))^s,$$

$$\int_a^b (t-a)^s \Delta t \leq (b-a)^{1-s}(h_2(b,a))^s.$$

Proof. Using Hölder's inequality on time scales, we get:

$$\int_a^b (b-t)^s \,\Delta t \leq \left(\int_a^b (b-t)\,\Delta t\right)^s \left(\int_a^b \Delta t\right)^{1-s}$$

$$= (b-a)^{1-s}\left(\int_b^a (t-b)\,\Delta t\right)^s$$

$$= (b-a)^{1-s}(h_2(a,b))^s$$

and

$$\int_a^b (t-a)^s \,\Delta t \leq \left(\int_a^b (t-a)\,\Delta t\right)^s \left(\int_a^b \Delta t\right)^{1-s}$$

$$= (b-a)^{1-s}(h_2(b,a))^s.$$

This completes the proof.

We recall here the following well-known inequality.

Theorem 2.11 [8] (Hölder's inequality) Let $a, b \in \mathrm{T}$, $a < b$. For rd -continuous functions $f, g: [a,b] \to \mathrm{R}$ we have

$$\int_a^b |f(t)g(t)|\Delta\tau \leq \left(\int_a^b |f(t)|^p \Delta\tau\right)^{1/p} \left(\int_a^b |g(t)|^q \Delta\tau\right)^{1/q},$$

where $p > 1$ and $1/p + 1/q = 1$.

3. Main Results

Theorem 3.1 Let $\alpha > 2$, $m-1 < \alpha < m$, $\nu = m - \alpha$, $f \in C_{rd}^{m}([a,b])$, $f^{\Delta^k}(a) = 0$, $k \in \{0,1,\ldots,m-1\}$, $|f^{\Delta^m}|$ is exponentially s -convex in the second sense function on $[a,b]$. Then

$$|B(t)| \leq \left(\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)} + \frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}\right) h_{\alpha+\nu}(t,a), \quad t \in [a,b], \tag{3.1}$$

and

$$|B(t)| \leq h_{\alpha+\nu-1}(t,a)(b-a)^{1-2s}\Big(\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}(h_2(a,b))^s + \frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}(h_2(b,a))^s\Big), \quad t \in [a,b]. \tag{3.2}$$

Proof. Since $|f^{\Delta^m}|$ is exponentially s -convex in the second sense on $[a,b]$, we have

$$|f^{\Delta^m}(t)| \leq \left(\frac{b-t}{b-a}\right)^s \frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)} + \left(\frac{t-a}{b-a}\right)^s \frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}, \quad t \in [a,b]. \tag{3.3}$$

1. Firstly, we will prove (3.1). We have

$$|B(t)| = \left|\int_a^t h_{\alpha-2}(t,\sigma(\tau))\big(\int_a^\tau h_\nu(\tau,\sigma(u))\, f^{\Delta^m}(u)\Delta u\big)\Delta\tau\right|$$

$$\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))|f^{\Delta^m}(u)|\Delta u\Delta\tau$$

$$\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))(\left(\tfrac{b-u}{b-a}\right)^s \tfrac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}$$

$$+ \left(\tfrac{u-a}{b-a}\right)^s \tfrac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)})\Delta u\Delta\tau$$

$$\leq \left(\tfrac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)} + \tfrac{|f^{\Delta^m}(b)}{e_\beta(b,x_0)}\right) \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))\Delta u\Delta\tau$$

$$= \left(\tfrac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)} + \tfrac{|f^{\Delta^m}(b)}{e_\beta(b,x_0)}\right) \int_a^t h_{\alpha-2}(t,\sigma(\tau)) h_{\nu+1}(\tau,a)\Delta\tau$$

$$= \left(\tfrac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)} + \tfrac{|f^{\Delta^m}(b)}{e_\beta(b,x_0)}\right) h_{\alpha+\nu}(t,a), \quad t \in [a,b].$$

2. Now, we will prove (3.2). Using Lemma 2.10, we have

$$|B(t)| \leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))(\left(\tfrac{b-u}{b-a}\right)^s \tfrac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}$$

$$+ \left(\tfrac{u-a}{b-a}\right)^s \tfrac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)})\Delta u\Delta\tau$$

$$\leq \left(\int_a^t h_{\alpha-2}(t,\sigma(\tau)) h_\nu(\tau,a)\Delta\tau\right) (\int_a^b (\left(\tfrac{b-u}{b-a}\right)^s \tfrac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}$$

$$+ \left(\tfrac{u-a}{b-a}\right)^s \tfrac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)})\Delta u)$$

$\leq h_{\alpha+\nu-1}(t,a)(\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}(b-a)^{1-2s}(h_2(a,b))^s$

$+\frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}(b-a)^{1-2s}(h_2(b,a))^s)$

$= h_{\alpha+\nu-1}(t,a)(b-a)^{1-2s}(\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}(h_2(a,b))^s$

$+\frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}(h_2(b,a))^s), \quad t \in [a,b].$

This completes the proof.

The next result can be stated as follows.

Theorem 3.2 Let $q \geq 1$, $\alpha \geq 2$, $m-1 < \alpha < m$, $\nu = m-\alpha$, $f \in C^m_{rd}([a,b])$, $|f^{\Delta^m}| \geq 1$ on $[a,b]$, $|f^{\Delta^m}|^q$ is exponentially s -convex in the second sense function on $[a,b]$, $f^{\Delta^k}(a) = 0$, $k \in \{0,1,\dots,m-1\}$. Then

$$|B(t)| \leq \left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)} + \frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right) h_{\alpha+\nu}(t,a), \quad t \in [a,b], \tag{3.4}$$

and

$$\begin{aligned} |B(t)| \quad &\leq \quad h_{\alpha+\nu-1}(t,a)(b-a)^{1-2s}(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}(h_2(a,b))^s \\ &+\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}(h_2(b,a))^s), \quad t \in [a,b]. \end{aligned} \tag{3.5}$$

Proof. Since $|f^{\Delta^m}|^q$ is exponentially s -convex in the second sense function on $[a,b]$, we have

$$|f^{\Delta^m}(t)|^q \leq \frac{(b-t)^s}{(b-a)^s}\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)} + \frac{(t-a)^s}{(b-a)^s}\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}, \quad t \in [a,b]. \tag{3.6}$$

1. Firstly, we will prove (3.4). We have

$|B(t)| = \left|\int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau h_\nu(\tau,\sigma(u)) f^{\Delta^m}(u)\Delta u\right)\Delta\tau\right|$

$\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))|f^{\Delta^m}(u)|\Delta u\Delta\tau$

$\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))|f^{\Delta^m}(u)|^q\Delta u\Delta\tau$

$\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))(\frac{(b-u)^s}{(b-a)^s}\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}$

$+\frac{(u-a)^s}{(b-a)^s}\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)})\Delta u\Delta\tau$

$\leq \left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)} + \frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right)\int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))\Delta u\Delta\tau$

$\leq \left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)} + \frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right)\int_a^t h_{\alpha-2}(t,\sigma(\tau)) h_{\nu+1}(\tau,a)\Delta\tau$

$= \left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)} + \frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right) h_{\alpha+\nu}(t,a), \quad t \in [a,b].$

2. Now, we will prove (3.5). Using Lemma 2.10, we find

$$|B(t)| \leq \int_a^t h_{\alpha-2}(t,\sigma(\tau)) \int_a^\tau h_\nu(\tau,\sigma(u))(\frac{(b-u)^s}{(b-a)^s}\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}$$

$$+\frac{(u-a)^s}{(b-a)^s}\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)})\Delta u\Delta\tau$$

$$\leq \left(\int_a^t h_{\alpha-2}(t,\sigma(\tau))h_\nu(\tau,a)\Delta\tau\right)(\frac{|f^{\Delta^m}(a)|^q}{(b-a)^s e_\beta(a,x_0)}\int_a^b (b-u)^s\Delta u$$

$$+\frac{|f^{\Delta^m}(b)|^q}{(b-a)^s e_\beta(b,x_0)}\int_a^b (u-a)^s\Delta u)$$

$$\leq h_{\alpha+\nu-1}(t,a)(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}(b-a)^{1-2s}(h_2(a,b))^s$$

$$+\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}(b-a)^{1-2s}(h_2(b,a))^s)$$

$$= h_{\alpha+\nu-1}(t,a)(b-a)^{1-2s}(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}(h_2(a,b))^s$$

$$+\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}(h_2(b,a))^s), \quad t\in[a,b].$$

This completes the proof.

Theorem 3.3 Let $\alpha \geq 2,\ m-1<\alpha<m,\ \nu=m-\alpha,\ p,q\geq 1,\ \frac{1}{p}+\frac{1}{q}=1,\ f\in C_{rd}^m([a,b]),\ |f^{\Delta^m}|$ is exponentially s -convex in the second sense function on $[a,b]$, $f^{\Delta^k}(a)=0,\ k\in\{0,1,\dots,m-1\}$. Then

$$|B(t)| \leq (b-a)^{\{1-2s\}/q}2^{1/q}G(\alpha,\nu,p,t,a)h_{\alpha-1}(t,a)(\frac{|f^{\Delta^m}(a)|}{(e_\beta(a,x_0))^{1/q}}(h_2(a,b))^{s/q}$$

$$+\frac{|f^{\Delta^m}(b)|}{(e_\beta(b,x_0))^{1/q}}(h_2(b,a))^{s/q}), \quad t\in[a,b],$$

where

$$G(\alpha,\nu,p,t,a)=\left(\int_a^t (h_\nu(t,\sigma(u)))^p\Delta u\right)^{1/p}, \quad t\in[a,b].$$

Proof. Since $|f^{\Delta^m}|^q$ is exponentially s -convex in the second sense function on $[a,b]$, we have the inequality (3.6). Then

$$|B(t)| = \left|\int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau h_\nu(\tau,\sigma(u))f^{\Delta^m}(u)\Delta u\right)\Delta\tau\right|$$

$$\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau))\int_a^\tau h_\nu(\tau,\sigma(u))|f^{\Delta^m}(u)|\Delta u\Delta\tau$$

$$\leq \int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau (h_\nu(\tau,\sigma(u)))^p\Delta u\right)^{1/p}\left(\int_a^\tau |f^{\Delta^m}(u)|^q\Delta u\right)^{1/q}$$

$$= \int_a^t h_{\alpha-2}(t,\sigma(\tau))G(\alpha,\nu,p,\tau,a)\left(\int_a^\tau |f^{\Delta^m}(u)|^q\Delta u\right)^{1/q}\Delta\tau$$

$$\leq G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\left(\int_a^\tau |f^{\Delta^m}(u)|^q\Delta u\right)^{1/q}\Delta\tau$$

$$\leq G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))(\int_a^\tau(\frac{(b-u)^s}{(b-a)^s}\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}$$

$+\frac{(u-a)^s}{(b-a)^s}\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)})\Delta u)^{1/q}\Delta\tau$

$\leq G(\alpha,\nu,p,t,a)\int_a^t h_{\alpha-2}(t,\sigma(\tau))\Delta\tau$

$\times\left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)(b-a)^s}\int_a^b(b-u)^s\Delta u+\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)(b-a)^s}\int_a^b(u-a)^s\Delta u\right)^{1/q}$

$\leq G(\alpha,\nu,p,t,a)((b-a)^{1-2s}\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}(h_2(a,b))^s$

$+(b-a)^{1-2s}\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}(h_2(b,a))^s)^{1/q}h_{\alpha-1}(t,a)$

$\leq(b-a)^{\{1-2s\}/q}2^{1/q}G(\alpha,\nu,p,t,a)h_{\alpha-1}(t,a)(\frac{|f^{\Delta^m}(a)|}{(e_\beta(a,x_0))^{1/q}}(h_2(a,b))^{s/q}$

$+\frac{|f^{\Delta^m}(b)|}{(e_\beta(b,x_0))^{1/q}}(h_2(b,a))^{s/q}),\quad t\in[a,b],$

where the last inequality results from $(x+y)^k\leq 2^k(x^k+y^k)$, $x,y\geq 0$, $k>0$. This completes the proof.

Theorem 3.4 Let $\alpha>2$, $m-1<\alpha<m$, $\nu=m-\alpha$, $f\in C_{rd}^m([a,b])$, $|f^{\Delta^m}|$ is exponentially s -convex in the second sense function on $[a,b]$. Then

$$|\Delta_a^{\alpha-1}f(t)|\leq\left(\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}+\frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}\right)h_{\nu+1}(t,a),\quad t\in[a,b].$$

Proof. Since $|f^{\Delta^m}|$ is exponentially s -convex in the second sense on $[a,b]$, we have the inequality (3.3). Then

$|\Delta_a^{\alpha-1}f(t)|=\left|\int_a^t h_\nu(t,\sigma(u))f^{\Delta^m}(u)\Delta u\right|$

$\leq\int_a^t h_\nu(t,\sigma(u))|f^{\Delta^m}(u)|\Delta u$

$\leq\int_a^t h_\nu(t,\sigma(u))\left(\left(\frac{b-t}{b-a}\right)^s\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}+\left(\frac{t-a}{b-a}\right)^s\frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}\right)\Delta u$

$\leq\left(\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}+\frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}\right)\int_a^t h_\nu(t,\sigma(u))\Delta u$

$=\left(\frac{|f^{\Delta^m}(a)|}{e_\beta(a,x_0)}+\frac{|f^{\Delta^m}(b)|}{e_\beta(b,x_0)}\right)h_{\nu+1}(t,a),\quad t\in[a,b].$

This completes the proof.

Theorem 3.5 Let $q\geq 1$, $\alpha>2$, $m-1<\alpha<m$, $\nu=m-\alpha$, $f\in C_{rd}^m([a,b])$, $|f^{\Delta^m}|\geq 1$ on $[a,b]$, $|f^{\Delta^m}|^q$ is exponentially s -convex in the second sense function on $[a,b]$. Then

$$|\Delta_a^{\alpha-1}f(t)|\leq\left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}+\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right)h_{\nu+1}(t,a),\quad t\in[a,b].$$

Proof. Since $|f^{\Delta^m}|^q$ is exponentially s -convex in the second sense on $[a,b]$, we have the inequality (3.6). Then

$|\Delta_a^{\alpha-1}f(t)|=\left|\int_a^t h_\nu(t,\sigma(u))f^{\Delta^m}(u)\Delta u\right|$

$\leq\int_a^t h_\nu(t,\sigma(u))|f^{\Delta^m}(u)|\Delta u$

$$\leq \int_a^t h_\nu\,(t,\sigma(u))|f^{\Delta^m}(u)|^q\Delta u$$

$$\leq \int_a^t h_\nu\,(t,\sigma(u))\left(\left(\frac{b-u}{b-a}\right)^s\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}+\left(\frac{u-a}{b-a}\right)^s\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right)\Delta u$$

$$\leq \left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}+\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right)\int_a^t h_\nu\,(t,\sigma(u))\Delta u$$

$$= \left(\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}+\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}\right)h_{\nu+1}(t,a),\quad t\in[a,b].$$

This completes the proof.

Theorem 3.6 Let $p,q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, $\alpha>2$, $m-1<\alpha<m$, $\nu=m-\alpha$, $f\in C_{rd}^m([a,b])$, $|f^{\Delta^m}|^q$ is exponentially s-convex in the second sense function on $[a,b]$. Then

$$|\Delta_a^{\alpha-1}f(t)|\leq (b-a)^{\{1-2s\}/q}2^{1/q}G(\alpha,\nu,p,t,a)(\frac{|f^{\Delta^m}(a)|}{(e_\beta(a,x_0))^{1/q}}(h_2(a,b))^{sq/}$$

$$+\frac{|f^{\Delta^m}(b)|}{(e_\beta(b,x_0))^{1/q}}(h_2(b,a))^{s/q}),\quad t\in[a,b],$$

where $G(\alpha,\nu,p,t,a)$, $t\in[a,b]$, is defined as in Theorem 3.3.

Proof. Because $|f^{\Delta^m}|^q$ is exponentially s-convex in the second sense function on $[a,b]$, the inequality (3.6) holds. Then

$$|\Delta_a^{\alpha-1}f(t)| = \left|\int_a^t h_\nu\,(t,\sigma(u))f^{\Delta^m}(u)\Delta u\right|$$

$$\leq \int_a^t h_\nu\,(t,\sigma(u))|f^{\Delta^m}(u)|\Delta u$$

$$\leq \left(\int_a^t(h_\nu(t,\sigma(u)))^p\Delta u\right)^{1/p}\left(\int_a^t|f^{\Delta^m}(u)|^q\Delta u\right)^{1/q}$$

$$= G(\alpha,\nu,p,t,a)\left(\int_a^t|f^{\Delta^m}(u)|^q\Delta u\right)^{1/q}$$

$$\leq G(\alpha,\nu,p,t,a)(\int_a^t((\frac{b-u}{b-a})^s\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}$$

$$+\left(\frac{u-a}{b-a}\right)^s\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)})\Delta u)^{1/q}$$

$$\leq G(\alpha,\nu,p,t,a)((b-a)^{1-2s}\frac{|f^{\Delta^m}(a)|^q}{e_\beta(a,x_0)}(h_2(a,b))^s$$

$$+(b-a)^{1-2s}\frac{|f^{\Delta^m}(b)|^q}{e_\beta(b,x_0)}(h_2(b,a))^s)^{1/q}$$

$$\leq (b-a)^{\{1-2s\}/q}2^{1/q}G(\alpha,\nu,p,t,a)(\frac{|f^{\Delta^m}(a)|}{(e_\beta(a,x_0))^{1/q}}(h_2(a,b))^{s/q}$$

$$+\frac{|f^{\Delta^m}(b)|}{(e_\beta(b,x_0))^{1/q}}(h_2(b,a))^{s/q}),\quad t\in[a,b].$$

This completes the proof.

Conclusion

In this study, we established new integral inequalities for exponentially sss-convex functions in the second sense within the framework of time scales. Utilizing the delta Riemann-Liouville fractional integral and fractional Taylor formula, we derived meaningful upper bounds for nnn-times rdrdrd-continuously Δ\DeltaΔ-differentiable functions. Our findings advance the understanding of time scale calculus by unifying discrete and continuous cases and provide a robust foundation for future investigations in fractional dynamic systems. These results not only enrich the theory of time scales but also open avenues for applications in various domains of mathematical analysis and its interdisciplinary connections.

Conflict of Interest

There is no conflict of interest.

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