New Coefficient Inequalities for Certain Subclasses of p- Valent Analytic Functions

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Abstract: The object of the present paper is to derive new coefficient inequalities for certain subclasses of $p-$ valent analytic functions defined in the open unit disk $\mathcal{U}$. Our results are generalized of the previous theorems given by J. Clunie and F.R. Keogh [1], by H. Silverman [3] and by M. Nunokawa *et al.* [2].

Keywords: Analytic functions, $p-$ valently starlike of order $\alpha,$ $p-$ valently convex of order $\alpha,$ coefficient inequalities.

1. INTRODUCTION

Let $\mathcal{A}_p$ denote the class of the form

$$f(z)=\sum_{n=p}^{\infty}a_n z^n, \quad \left(a_p=1,\ p,\ n\in\mathbb{N}=\left\{1,2,...\right\}\right), \quad (1.1)$$

which are analytic and $p-$ valent in the open disk $\mathcal{U}=\left\{z\in\mathbb{C}\ :\ \left|z\right|<1\right\}$. We note that $\mathcal{A}_1=\mathcal{A}$.

A function $f\in\mathcal{A}_p$ is said to be $p-$ valently starlike of order $\alpha\left(0\leq\alpha<p\right)$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, \quad \left(z\in\ \mathcal{U}\right).$$

The class of all such functions are denote by $\mathcal{S}_p^*(\alpha)$. Here, $\mathcal{S}_1^*\left(\alpha\right)=\mathcal{S}^*\left(\alpha\right)$ and $\mathcal{S}^*\left(0\right)=\mathcal{S}^*$ are the classes of starlike function of order $\alpha\left(0\leq\alpha<1\right)$ and starlike function, respectively. On the other hand, a function $f\in\mathcal{A}_p$ is said to be $p-$ valently convex of order $\alpha\left(0\leq\alpha<p\right)$ if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha, \quad \left(z\in\ \mathcal{U}\right).$$

Let $\mathcal{C}_p\left(\alpha\right)$ denote the class of all those functions. Also $\mathcal{C}_1\left(\alpha\right)=\mathcal{C}\left(\alpha\right)$ and $\mathcal{C}\left(0\right)=\mathcal{C}$ are the classes of convex function of order $\alpha\left(0\leq\alpha<1\right)$ and convex function, respectively.

Clunie and Keogh [1] (also Silverman [3]) have proved the following results: If $f(z)\in\mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}n\left|a_n\right|\leq 1,$$

then $f(z)$ is univalent and starlike in $\mathcal{U}$. If $f(z)\in\mathcal{A}$ satisfies

$$\sum_{n=2}^{\infty}n^2\left|a_n\right|\leq 1,$$

then $f(z)$ is univalent and convex in $\mathcal{U}$.

Nunokawa *et al*. [2] have proved the following results: Let $f(z)$ be of the class $\mathcal{A}$ and $\max_{n\geq 1}\left|a_n\right|=t\left|a_t\right|$. If $f(z)\in\mathcal{A}$ satisfies

$$\sum_{n=1,\ n\neq t}^{\infty}\left(\left|n-t\right|+t\right)\left|a_n\right|\leq t\left|a_t\right|,$$

then $f(z)$ is univalent and starlike in $\mathcal{U}$. Let $f(z)$ be of the class $\mathcal{A}$ and $\max_{n\geq 1}n^2\left|a_n\right|=t^2\left|a_t\right|$. If $f(z)\in\mathcal{A}$ satisfies

$$\sum_{n=1,\ n\neq t}^{\infty}n\left(\left|n-t\right|+t\right)\left|a_n\right|\leq t^2\left|a_t\right|,$$

then $f(z)$ is univalent and convex in $\mathcal{U}$.

In the present investigation, we consider new coefficient inequalities for functions $f(z)$ to be $p-$

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valently starlike of order α and $p-$ valently convex of order α in $\mathcal{U}$.

2. COEFFICIENT INEQUALITIES

Our first result for functions $f(z)$ to be $p-$ valently starlike of order α in $\mathcal{U}$ is contained in the following Theorem 2.1.

Theorem 2.1. Let $f(z)$ be in the class $\mathcal{A}_p$ and

$$\max_{n\geq p} n\left|a_n\right| = \left(t+p-1\right)\left|a_{t+p-1}\right|.$$

If $f(z)\in\mathcal{A}_p$ satisfies the following inequality

$$\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left(\left|n-t-p+1\right|+t+p-1+\alpha\right)\left|a_n\right| \leq \left(t-p+1+\alpha\right)\left|a_{t+p-1}\right|, \tag{2.1}$$

then $f(z)$ is $p-$ valently starlike of order α in $\mathcal{U}$.

Proof: Applying the maximum principle of analytic functions, the following inequality is hold on $|z|=1$

$$\left|zf'(z)-tf(z)-\left(p-1\right)f(z)\right|-\left|tf(z)\right|-\left|\left(p-1\right)f(z)\right|+\left|\alpha f(z)\right|$$

$$=\left|\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left(n-t-p+1\right)a_nz^n\right|-t\left|\sum_{n=p}^{\infty}a_nz^n\right|-\left(p-1\right)\left|\sum_{n=p}^{\infty}a_nz^n\right|+\alpha\left|\sum_{n=p}^{\infty}a_nz^n\right|$$

$$\leq\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left|n-t-p+1\right|\left|a_n\right|\left|z^n\right|-t\left(\left|a_{t+p-1}\right|\left|z\right|^{t+p-1}-\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left|a_n\right|\left|z^n\right|\right)$$

$$-\left(p-1\right)\left(\left|a_{t+p-1}\right|\left|z\right|^{t+p-1}-\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left|a_n\right|\left|z^n\right|\right)+\alpha\left(\left|a_{t+p-1}\right|\left|z\right|^{t+p-1}-\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left|a_n\right|\left|z^n\right|\right)$$

$$=\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left(\left|n-t-p+1\right|+t+p-1+\alpha\right)\left|a_n\right|-\left(t-p+1+\alpha\right)\left|a_{t+p-1}\right|\leq 0.$$

Therefore, it follows that the following inequality

$$\left|\frac{zf'(z)}{f(z)}-t-\left(p-1\right)\right|\leq t+\left(p-1\right)-\alpha$$

holds for all $z\in\mathcal{U}$. This shows that $f(z)$ is $p-$ valently starlike of order α in $\mathcal{U}$.

If we take $\alpha=0$ in the Theorem 2.1., we get the following corollary.

Corollary 2.2. Let $f(z)$ be in the class $\mathcal{A}_p$ and

$$\max_{n\geq p} n\left|a_n\right| = \left(t+p-1\right)\left|a_{t+p-1}\right|.$$

If $f(z)\in\mathcal{A}_p$ satisfies the following inequality

$$\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty}\left(\left|n-t-p+1\right|+t+p-1\right)\left|a_n\right|\leq\left(t-p+1\right)\left|a_{t+p-1}\right|,$$

then $f(z)$ is $p-$ valently starlike in $\mathcal{U}$.

For $p=1$ in the Theorem 2.1., we have the following corollary.

Corollary 2.3. Let $f(z)$ be in the class $\mathcal{A}$ and

$$\max_{n\geq 1} n\left|a_n\right| = t\left|a_t\right|.$$

If $f(z)\in\mathcal{A}$ satisfies the following inequality

$$\sum_{n=1,n\neq t}^{\infty}\left(\left|n-t\right|+t+\alpha\right)\left|a_n\right|\leq\left(t+\alpha\right)\left|a_t\right|,$$

then $f(z)$ is starlike of order α in $\mathcal{U}$.

Next, we derive the coefficient condition for functions $f(z)$ to be $p-$ valently convex of order α in $\mathcal{U}$ is contained in the Theorem 2.4 as given below.

Theorem 2.4. Let $f(z)$ be in the class $\mathcal{A}_p$ and

$$\max_{n\geq p} n^2\left|a_n\right| = \left(t+p-1\right)^2\left|a_{t+p-1}\right|.$$

If $f(z)\in\mathcal{A}_p$ satisfies the following inequality

$$\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} n\left(\left|n-t-p+1\right|+t+p-1+\alpha\right)\left|a_n\right| \leq\left(t+p-1\right)\left(t+p-1-\alpha\right)\left|a_{t+p-1}\right|, \tag{2.2}$$

then $f(z)$ is $p-$ valently convex of order α in $\mathcal{U}$.

Proof: Applying the maximum principle of analytic functions, the following inequality is hold on $|z|=1$

$$\left|zf''(z)+f'(z)-tf'(z)-\left(p-1\right)f'(z)\right|-\left|tf'(z)\right|-\left|\left(p-1\right)f'(z)\right|+\left|\alpha f'(z)\right|$$

$$= \left| \sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} \left[n\left(n-t-p+1\right) \right] a_n z^{n-1} \right| - t \left| \sum_{n=p}^{\infty} n a_n z^{n-1} \right|$$

$$-\left(p-1\right)\left| \sum_{n=p}^{\infty} n a_n z^{n-1} \right| + \alpha \left| \sum_{n=p}^{\infty} n a_n z^{n-1} \right|$$

$$\leq \sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} n\left|n-t-p+1\right|\left|a_n\right|\left|z^{n-1}\right| - t$$

$$\left(\left(t+p-1\right)\left|a_{t+p-1}\right|\left|z\right|^{t+p-1} - \sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} n\left|a_n\right|\left|z^{n-1}\right| \right)$$

$$-\left(p-1\right)\left(\left(t+p-1\right)\left|a_{t+p-1}\right|\left|z\right|^{t+p-1} - \sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} n\left|a_n\right|\left|z^{n-1}\right| \right)$$

$$+\alpha\left(\left(t+p-1\right)\left|a_{t+p-1}\right|\left|z\right|^{t+p-1} - \sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} n\left|a_n\right|\left|z^{n-1}\right| \right)$$

$$= \sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} n\left(\left|n-t-p+1\right| + t+p-1+\alpha\right)\left|a_n\right|$$

$$-\left(t+p-1\right)\left(t+p-1-\alpha\right)\left|a_{t+p-1}\right| \leq 0.$$

Therefore, it follows that the follwing inequality

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right) - t - \left(p-1\right)\right| \leq t + \left(p-1\right) - \alpha$$

holds for all $z \in \mathcal{U}$. This shows that $f(z)$ is $p-$ valently convex of order α in $\mathcal{U}$.

By taking $\alpha = 0$ in the Theorem 2.4, we get the following corollary.

Corollary 2.5. Let $f(z)$ be in the class $\mathcal{A}_p$ and

$$\max_{n\geq p} n^2\left|a_n\right| = \left(t+p-1\right)^2\left|a_{t+p-1}\right|.$$

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If $f(z) \in \mathcal{A}_p$ satisfies the following inequality

$$\sum_{\substack{n=p,\\ n\neq t+p-1}}^{\infty} n\left(\left|n-t-p+1\right| + t+p-1\right)\left|a_n\right| \leq \left(t+p-1\right)^2\left|a_{t+p-1}\right|,$$

then $f(z)$ is $p-$ valently convex in $\mathcal{U}$.

By taking $p = 1$ in the Theorem 2.4, we get the following corollary.

Corollary 2.6. Let $f(z)$ be in the class $\mathcal{A}$ and

$$\max_{n\geq 1} n^2\left|a_n\right| = t^2\left|a_t\right|.$$

If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\sum_{n=1, n\neq t}^{\infty} n\left(\left|n-t\right| + t+\alpha\right)\left|a_n\right| \leq t\left(t-\alpha\right)\left|a_t\right|,$$

then $f(z)$ is convex of order α in $\mathcal{U}$.

Remark 2.7. By considering some special values for the parameters α, p and t, we can deduce the following results.

In the Theorem 2.1. and Theorem 2.4., for $p = 1$ and $\alpha = 0$, we get the result given by Nunokawa *et al*. [2].

In the Theorem 2.1. and Theorem 2.4., for $p = 1$, $\alpha = 0$ and $t = 1$, we obtain the result given by Clunie and Keogh [1] (also Silverman [3]).

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