Convergence of the Galerkin Method for Nonlinear Dynamics of the Continuous Structure

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Abstract: The Galerkin truncation method is a powerful method for nonlinear dynamics analysis, and has been widely used to discretize the spatial differential operator. Due to more and more new fields of application, the research interest on the Galerkin method is still high today. In this paper, research on the convergence of Galerkin method for nonlinear dynamics of the continuous structure is thoroughly reviewed. At the beginning, the Galerkin method is briefly introduced. Then, the paper reviews the application of the truncation method. This paper also sums up the comparative study on the Galerkin method with other methods, such as the finite difference method (FDM), the finite element method (FEM), and the multiple time scales method. In the investigations concerning the convergence of the Galerkin method, this paper summarizes recent studies on nonlinear dynamics of the axially moving systems, the continua on the nonlinear foundation, and the belt-pulley systems. Finally, the truncation terms of Galerkin method for the continuous structure's nonlinear dynamics analysis is suggested for the future research applications.

Keywords: Galerkin method, nonlinear dynamics, nonlinear vibration, convergence, continuous structure.

1. INTRODUCTION

The dynamical problems of the continuous structure, which is described by partial differential equations with boundary conditions, usually display nonlinear characters and the solution becomes difficult. The Galerkin method is a powerful tool for such cases. The Galerkin method is usually credited to the Russian mathematician Boris Galerkin for finding the approximate solution of an operator equation [1]. The approach is used to convert a continuous operator problem to a discrete problem in the area of approximate numerical analysis. In principle, the Galerkin method is the equivalent of applying the method variation of parameters to a function space, by converting the equation to a weak formulation. Applying the Galerkin method, the space is cha-racterized with a finite set of basis functions by using some constraints on the function space. Furthermore, the differential equations are simplified into linear equations to solve problems. There is a general approach to approximate methods, which includes the projection methods, the FDM and other approximate methods which are generalizations of the Galerkin method. Moreover, the nowadays widely used the FEM is also a special case of the Galerkin method.

For dynamical problems of continuum by using the Galerkin method, one assumes the distributed coordinates as

$$u(x,t)=\sum_{j=1}^{n} q_j(t)\phi_j,\ j=1,2,\ldots,n \tag{1}$$

where x is the neutral axis coordinate of the continuum, $u(x,t)$ is the displacement of the continuum at x and time t, $q_j(t)$ (j=1,2…n) is a set of generalized coordinates or modal coordinates for the continuum, $\phi_j(x)$ (j=1,2…n) is the trial function, which satisfies or partial satisfies the given boundary conditions. The weight functions, $w_i(x)$ (i=1,2…n), can be the same as the trail functions. With the concepts of the inner product and orthogonality, a set of n-order linear or nonlinear ordinary differential equations are obtained. Then, the continuous system is idealized as a multi-degree-of-freedem system with generalized coordinates $\{\phi\}$ [2]. The Galerkin method is called the Bubnov–Galerkin method if the coordinate and the projection systems are identical [3]. In addition, the Galerkin method is sometimes called the Petrov–Galerkin method in Hilbert spaces. For analyzing the complex mechanical models of nonhomogeneous structures, a hybrid Wentzel-Kramer-Brillouin-Galerkin (or WKB-Galerkin) is employed [4]. This paper presents an easy-to-follow tutorial of the application of the Galerkin method for the nonlinear dynamics of the continuous structure.

2 SHORT REVIEW OF THE APPLICATION OF THE GALERKIN METHOD

The Galerkin method is frequently used in engineering to estimate the static and dynamic behavior of continuous structures with quite good accuracy and convergence. The complex nonlinear behavior of an axially loaded cylindrical shell was

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studied [5]. A derivation of the finite element equations for vibration of a thermopiezoelectric infinite cylinder was presented [6]. The Galerkin approach is carried out for the stochastic analysis of cables in turbulent wind through a reduced-order model containing both mechanical and aerodynamic non-linearities [7]. A relatively small number of modes are used to investigate the global non-linear behavior and the stability of a thin-walled fluid-filled cylindrical shell under lateral pressure and axial loads [8]. The eigenvalues of a fluid conveying tube were obtained [9]. By a simple mechanical model, it was also demonstrated that the Galerkin method may fail completely, if the energy dissipation is governed by the stable modes. For investigating nonlinear and viscoelastic behaviour, Galerkin's method is used to derive modal amplitude equations for a pinned-pinned beam interacting with polyurethane foam foundation [10]. Cheung and Zhou derived the eigenfrequency equation of a flexible thin plate placed [11]. The regular and chaotic vibrations of an axially moving viscoelastic string were presented [12]. Nonlinear forced vibration of a viscoelastic buckled beam [13] and of a viscoelastic pipe conveying fluid around curved equilibrium due to the supercritical flow [14] subjected to primary resonance in the presence of Two-to-One internal resonance is investigated.

The Galerkin method is modified and improved by many researchers. Gorman and Ding obtained the accurate free vibration analysis of symmetric [15] and antisymmetric [16] angle-ply laminated rectangular plates and the thick Mindlin plate [17] by the Superposition-Galerkin method. The authors enumerated the advantages of the superposition-Galerkin method over the traditional superposition method, such as the hyperbolic functions no longer appear in the analysis. Combining the polynomial shape functions of conventional finite element analysis with Galerkin orthogonal functions, the Galerkin element method is applied to the vibration of damped sandwich beam structures [18] and damped sandwich plates [19]. The numerical results showed that only a few elements are required to obtain even high-frequency modal parameters with very good accuracy. The Galerkin method in conjunction with natural coordinates had been presented as the basis for the dynamic analysis of general simply supported plates with intermediate line or point supports [20]. To analyze the natural frequencies of composite laminates of complicated shape, Chen *et al*. proposed an efficient and robust element free Galerkin method [21]. Amabili *et al*. found that the proper orthogonal decomposition in conjunction with the Galerkin approach permits a lower-dimensional model [22]. Natural frequencies, mode shapes and dynamic tensions of an elastic cable carrying an attached mass are obtained by using the Galerkin method with anti-derivatives of Daubechies wavelets [23]. The study showed that the frequencies and modes are concerned the Fourier and wavelet solutions are generally in good agreement. As far as the dynamic tension is concerned, the numerical results gave the indication of presence of error in the Fourier solution. The Bubnov–Galerkin method was used to study the thermo-mechanical vibration of the single-walled carbon nanotube embedded in a Winkler-type elastic medium [24].

3. SHORT REVIEW OF THE COMPARISON WITH OTHER METHODS

For determining the accuracy and the efficiency in the applications, the Galerkin method had been compared with some other methods. Based on the chaotic vibrations of circular cylindrical shells, the conventional Galerkin method is compared with the proper orthogonal decomposition method in conjunction with the Galerkin approach [25]. The numerical results showed that the conventional Galerkin method is more “robust”. The comparison between the generalized Galerkin method and the FEM analysis shows a pretty good match until the primary postbuckling equilibrium path comes close to the secondary bifurcation point, where ANSYS fails to converge [26]. Cepon and Boltezar preformed a quantitative comparison between three approximate methods, namely, the Galerkin finite-element method, the Galerkin method, and the finite-difference method [27]. Considering a comparison of the approximate solutions with the exact solution, one can assume that the Galerkin method gives a poor prediction of the dynamic response of an axially moving continuum at high velocities. The nonlinear behavior of a transversally excited, buckled pinned-pinned beam is studied by higher order single-mode as well as multimode Galerkin discretizations and is verified by finite element analyses. It is concluded that the difference in the dynamic response of the FEM analyses and the Galerkin analyses with higher order approximations becomes very small [28]. Fazzolari and Carrera compared the Rayleigh-Ritz, Galerkin and Generalized Galerkin methods for accurate buckling and vibration analysis of anisotropic laminated composite plates [29].

The attention was paid especially to improve the Galerkin method for dynamic analysis of the elastic system. A modified Galerkin approach was presented for free vibrations of delaminated unidirectional sandwich panels with a transversely flexible core [30]. The numerical results compare very well with those of the FEM results. The periodic nonlinear vibration of light axially moving band is solved by the Fourier–Galerkin–Newton method [31] and verified by experimental results [32]. Amirani *et al*. found that the results of the element free Galerkin method showed good agreement with those obtained by the FEM for free vibration analysis of sandwich beam with functionally graded core [33]. A mesh-free Galerkin method for the free vibration analysis of corrugated plates was proposed by Liew *et al*. [34]. The results showed a good agreement with the solutions that are derived with the FEM commercial software ANSYS. The Element-Free Galerkin method was used for free vibration analysis of point supported nonhomogeneous moderately thick plates resting on a two-parameter type elastic foundation [35]. It was found that the Element-Free Galerkin had very good agreements with available literature even with small number of nodes. Peng *et al.* proposed a meshfree Galerkin method for analyzing free vibration of corrugated-core sandwich plates [36]. The results showed a good agreement with the solutions that are derived with the ANSYS software. For analyzing the elasto-plastic problem of the moderately thick plate, a meshless local Petrov–Galerkin method was used with a radial basis function coupled with a polynomial basis function [37]. Numerical results show a good agreement compared with the results obtained using the ANSYS and the literature. Based on the moving Kriging interpolation technique, a meshless local Petrov-Galerkin approach achieved more accurate numerical solutions than the FEM for nonlinear bending problems of functionally graded plates in thermal environments but require less CPU time [38]. Through the bending and free vibration analysis of composite plates, it is observed that the meshless natural neighbor Galerkin method solutions are closer to the analytical solutions and slightly higher than solutions that are obtained using the FEM [39]. The meshless local Petrov-Galerkin method is implemented to analyze the free vibration and axial buckling characteristics of single-walled carbon nanotubes [40]. The numerical results are shown to be in good agreement with the exact solution.

4. THE CONVERGENCE OF THE GALERKIN METHOD

The convergence of solutions is achieved when the variation of displacements of the beam with a number of coordinate functions is sufficiently small. For investigating the nonlinear responses of buckled beams to primary-resonance excitations [41] and subharmonic-resonance excitations [42], a multi-mode Galerkin discretization found that using a single-mode approximation leads to quantitative and qualitative errors in the static and dynamic behaviors and cannot predict some of these nonlinear phenomena, such as Hopf bifurcation. The theoretical results are in good qualitative agreement with the experiment results. Wang analyzed the dynamics of a finite inextensible beam with an attached accelerating mass, and found that there was negligible difference between the results for 20-term and 30-term truncation [43]. Eshmatov and Khodjaev studied the numerical convergence of the Bubnov-Galerkin method for the non-linear vibration and dynamic stability of a viscoelastic cylindrical panel with concentrated mass [44]. Results show that a further increase in the number of components, the retained modes are more than seven, does not essentially influence the amplitude of vibrations, and the boundary terms influence the results. Nonlinear vibrations of viscoelastic orthotropic rectangular plates [45] and viscoelastic composite cylindrical panels [46] were studied by retaining the first five harmonics and the first seven harmonics, respectively. The numerical convergence study showed that a further increase in the number of components does not greatly influence the amplitude of vibration. For studying transverse vibrations of double-beam systems with viscoelastic inner layer, the Galerkin-type approximations are developed by Palmeri and Adhikari [47]. Numerical examples show that the 6-term Galerkin-type approach converges faster than a classical finite-element modelling.

The references showed that the convergence of the Galerkin method was influenced by the basis function. Based on the non-linear dynamics of continuous elastic systems, it can be concluded that the proper orthogonal decomposition basis is not more efficient than the Hilbertian basis [48]. For analyzing in-plane vibrations of flat-sag suspended cables carrying an array of moving oscillators with arbitrarily varying velocities, it has been found that such series exhibits poor convergence because of the singularities associated with the moving oscillators [49]. Furthermore, the study showed that the convergence of the series expansion is improved through the introduction of the so-called “quasi-static” solution. A spectral-Tchebychev technique was presented for obtaining the spatially discretized equations of motion

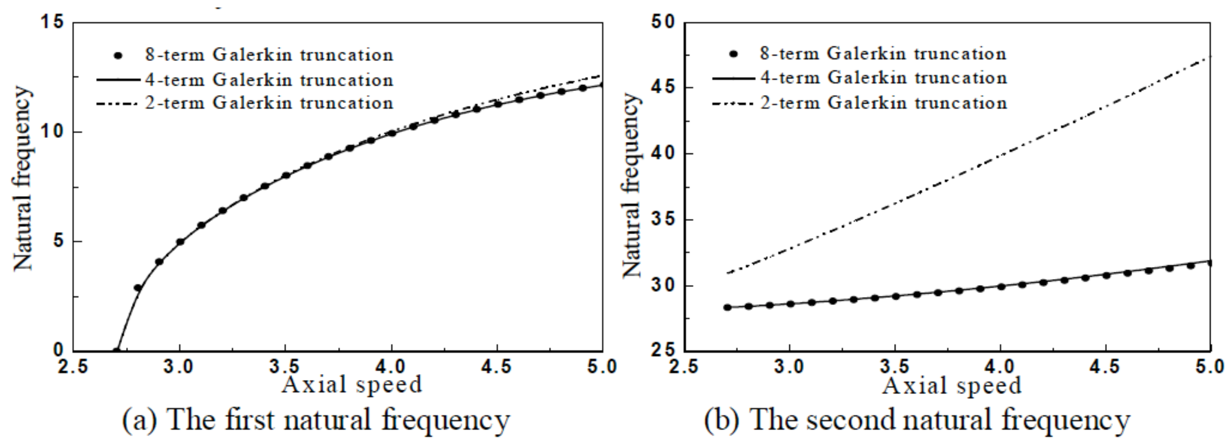


Figure 1: Comparison between 2-term, 4-term and 8-term Galerkin truncation results for the planar vibration.

[50]. The authors found that only a small number of polynomials are sufficient to obtain the machine-precision accuracy. For the free vibration analysis of doubly curved shallow shells, the study of Mochida *et al*. showed very good convergence rate for the fundamental natural frequencies with only five terms of the driving coefficients [51].

5. REVIEW OF SELECTED WORKS PUBLISHED BY THE AUTHOR

5.1. Convergence Studies on the Axially Moving Systems

Axially moving systems are extensively studied because they can model many engineering devices such as elevator cables, belt saws, paper sheet and web processes, fiber winding and power transmission band. The axially moving speed greatly affects the dynamic behavior of the system. Above a certain critical velocity, the first natural frequency of the system becomes zero and the straight configuration of equilibrium becomes unstable, with multiple coexisting equilibrium positions. For analyzing the dynamic behavior in the sub and super-critical speed ranges, Pellicano and Vestroni used the Galerkin method to discretize an axially moving beam subjected to an axial transport of mass [52]. The authors found that eight terms of the series expansion are sufficient to describe the response correctly. For motion about each bifurcated solution, the nonlinear governing equations are cast in the standard form of continuous gyroscopic systems by introducing a coordinate transform. The first two natural frequencies are obtained for the transverse vibration [53] and the coupled planar vibration [54] *via* the Galerkin method. As shown in Figure **1**, the 2-term Galerkin truncation for the natural frequency for axially moving beams in the supercritical range is bigger than the 4-term ones and the difference increase with the growth of axial speed, and the 4-term Galerkin method yields rather accurate results.

It is assumed that the excitation of the forced vibration is spatially uniform and temporally harmonic. For an axially moving beam constituted by the Kelvin model, the steady-state response of the primary resonance is analyzed *via* the Galerkin method in the supercritical speed range [55]. Figure **2** shows that 1-term Galerkin truncation yields the convincing primary

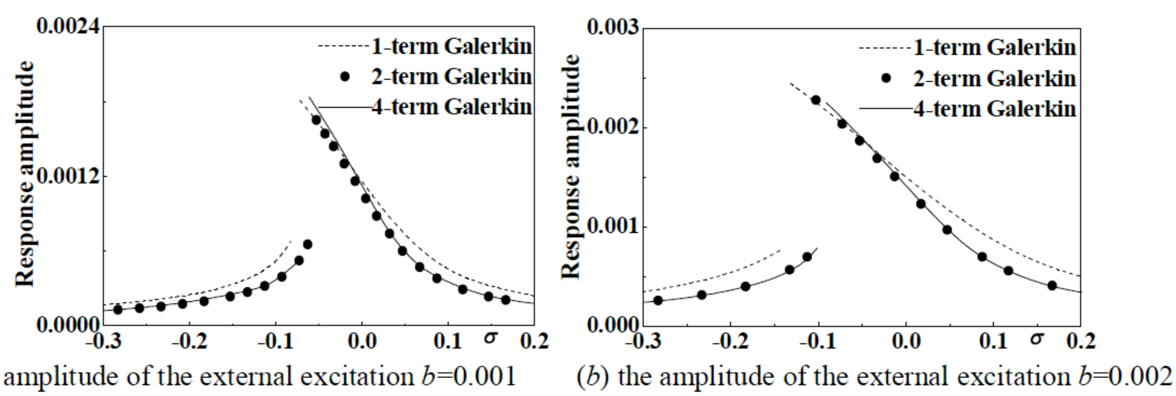


Figure 2: Comparisons among the 1-term, 2-term and 4-term Galerkin truncation for the primary resonance.

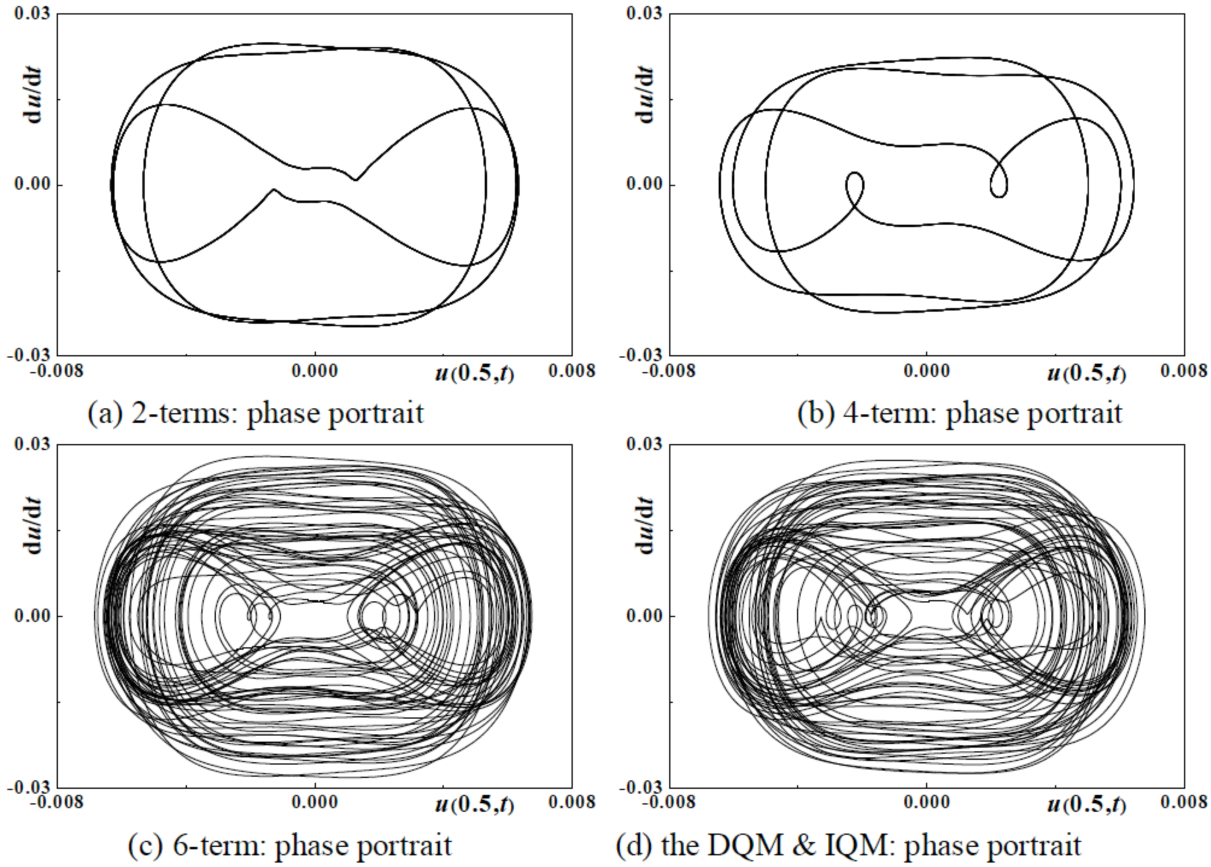


Figure 3: The nonlinear dynamics of an axially accelerating viscoelastic Euler-Bernoulli beam by using the Galerkin method with various truncations and DQM & IQM.

resonance. However, the results of the 2-term truncation are closer to those of the 4-term truncation than the 1-term Galerkin method. σ is the detuning parameter, which is introduced to quantify the deviation of the frequency of the external excitation from the fundamental frequency.

The bifurcation and chaos of an axially accelerating viscoelastic Euler-Bernoulli beam are investigated by using the Galerkin method with various truncations and the differential and integral quadrature methods (DQM & IQM) in the supercritical regime [56]. The speed of the axially moving beam is assumed to be comprised of a constant mean value along with harmonic fluctuations. By comparison with the DQM & IQM, the numerical results showed that the 2-term, 4-term, and 6-term Galerkin truncation and the DQM & IQM all predict that the chaotic motion and the periodic motion exchange alternately. As shown in Figure **3**, there is qualitative disagreement among the different term truncated system, and the 6-term Galerkin truncation and the DQM & IQM predict similar motion forms.

The steady-state periodic response and the chaos and bifurcation of an axially accelerating viscoelastic Timoshenko beam werestudied in the sub and super-critical speed ranges [57]. The Galerkin truncation is applied to discretize the governing equations into a set of nonlinear ordinary differential equations. The convergence of the Galerkin truncation was investigated. From the observation of Figure **4**, one can find that there are significant differences between the results of the stable steady-state periodic response based on the 2-term truncation and the other two truncations. Furthermore, the 2-term, 4-term, and 6-term Galerkin truncation all deliver that the periodic motion and the chaotic motion exchange alternately with the varying axial mean speed, but there are certain difference for bifurcation point prediction.

5.2. Convergence Studies on the Continua on the Nonlinear Foundation

Elastic beams resting on different types of foundations are extensively investigated because they can model many mechanical structures such as bridges, roads or airport pavements, railway enginee-ring equipments, transversally supported pipelines. The dynamic response problem of elastic beams lying on foundations displays nonlinear and viscous characters,

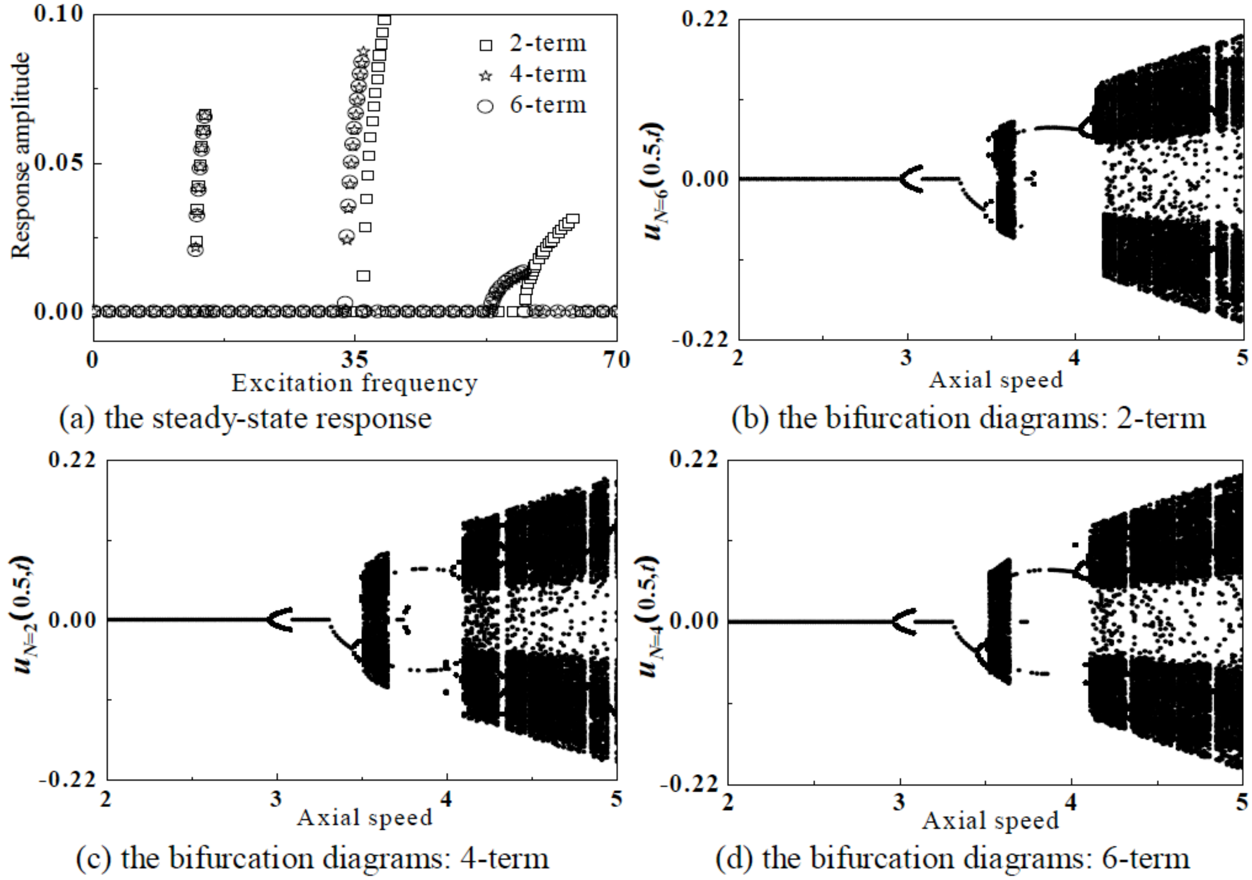


Figure 4: The nonlinear dynamics of an axially accelerating viscoelastic Timoshenko beam by using the Galerkin method with various truncations.

and the solution becomes difficult. The Galerkin method is a common tool for dealing with dynamical problems for such cases. Convergences of Galerkin truncation for dynamical response of beams on nonlinear foundations [58] and Timoshenko beams resting on a six-parameter foundation [59] under a moving load were studied based on the asphalt pavement resting on soft soil foundation moving the vehicle. In Figure **5**, *l* is the length of the elastic beam resting on foundations, *n* is the truncation term. The numerical results in Figure **5** demonstrated that the 50–term Galerkin method is not accurate enough for the dynamical response analysis the asphalt pavement on soft soil foundation running the vehicle, and there are discernible differences between the results of the 75–term and the 150–term Galerkin method. The comparisons predict that the 150–term Galerkin method yields rather accurate results. The investigation also found that the convergence of the Galerkin truncation can be predicted by the natural frequencies, the property of slower growth in the natural frequency of beam causes lack of convergence. Furthermore, the convergence accelerates with growing of the modulus of elasticity of the beams, the nonlinear foundation parameters, the span length and width of the pavement, but slows down with greater linear foundation parameters and damping coefficient.

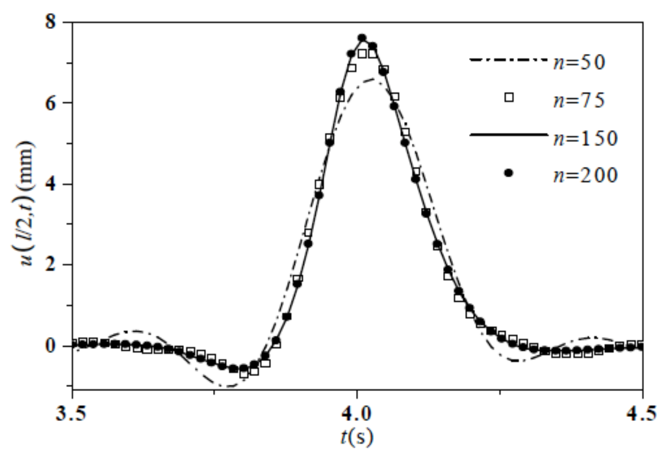
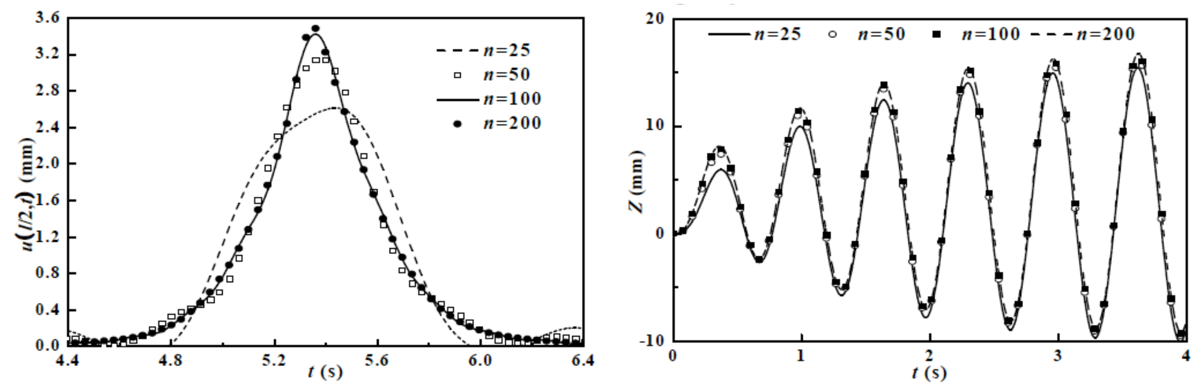


Figure 5: Effects of the Galerkin truncation terms on the vertical displacements of the pavement's midpoint.

The coupled nonlinear vibration of vehicle–pavement system is investigated by using the Galerkin method [60]. The pavement is modeled as a Timoshenko beam resting on a six–parameter



(a) on the vertical displacement of the beam's midpoint (b) on the vertical displacement of the vehicle body

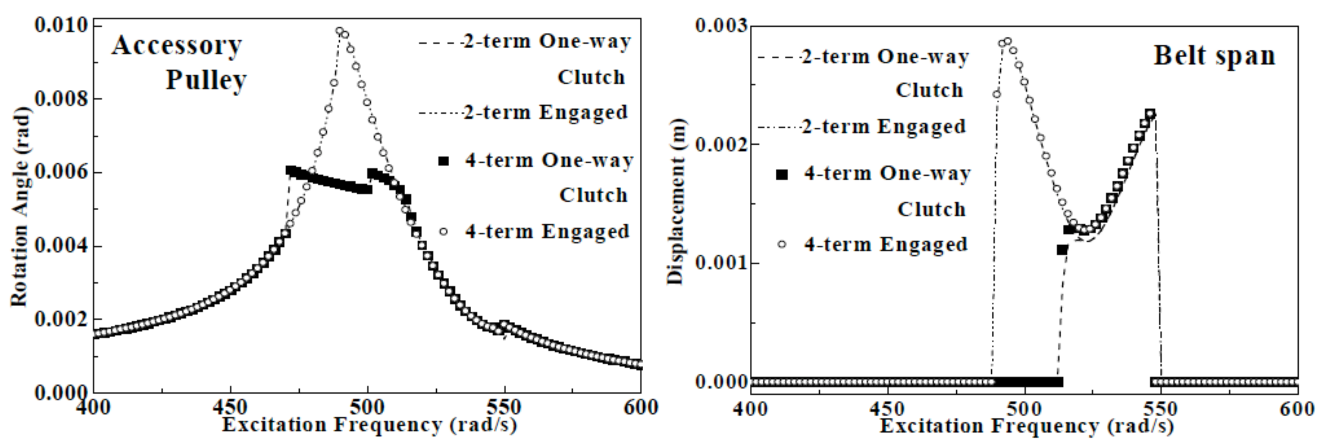
Figure 6: Convergence of the Galerkin method for the coupled nonlinear vibration of vehicle–pavement system.

foundation. Moreover, the vehicle is simplified as a spring–mass–damper oscillator. The Galerkin method was applied to discretize the nonlinear governing equation. The numerical results in Figure **6** showed that the dynamic response of the Timoshenko beam subjected to a moving oscillator needs more than 50 terms of the modal truncation. Furthermore, the convergence of the Galerkin truncation for Timoshenko beams on foundations is slightly slower than the Euler–Bernoulli beam.

5.3. Convergence Studies on the Belt-Pulley Systems

Pulley-belt systems, involving a flexible belt and several rigid pulleys, are widely applied to transmit power between rotational machine elements. For eliminating the effects of the greater weight accessory on the belt-drive systems, it is necessary to have a one-way clutch between the driven pulley and the accessory shaft. The nonlinear steady–state response of a belt-drive system with a one-way clutch was studied [61]. The derived coupled discrete-continuous nonlinear equations consist of integro–partial–differential equations and piece-wise ordinary differential equations. The resonance responses of the coupled belt-drive system are determined by using the Galerkin method. Furthermore, the results of the 2-term and 4-term Galerkin truncation are compared to determine the numerical convergence. As shown in Figure **7**, the two-term Galerkin method provides convergent numerical result in predicting the steady-state response of the pulley-belt system considering a string model for the belt.

Based on the non-trivial equilibrium, the steady-state periodic response of belt-drive system with a one-way clutch and belt flexural rigidity was studied *via* the Galerkin truncation as well as the DQM & IQM [62]. The belt spans were modeled as axially moving viscoelastic beams. New nonlinear governing equations are derived by introducing a coordinate transform based on the non-trivial equilibrium. The influences of the truncation terms, such as 6-term, 8-



(a) the comparison of the driven pulley; (b) the comparison of the belt span

Figure 7: The comparisons of the steady-state response between 2-term and 4-term Galerkin truncation.

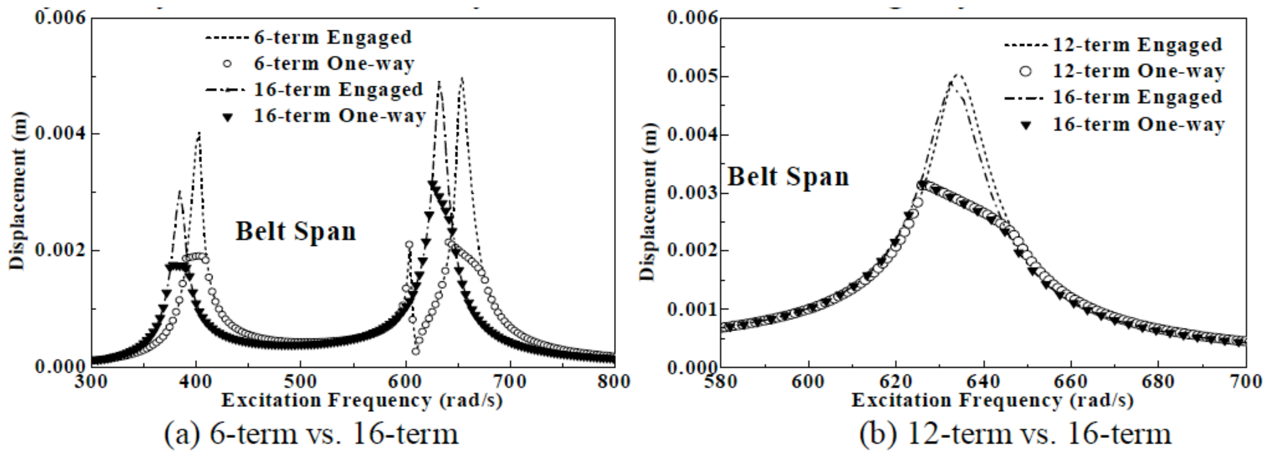


Figure 8: The comparisons of the steady-state response for different truncation terms: at a relatively low frequency.

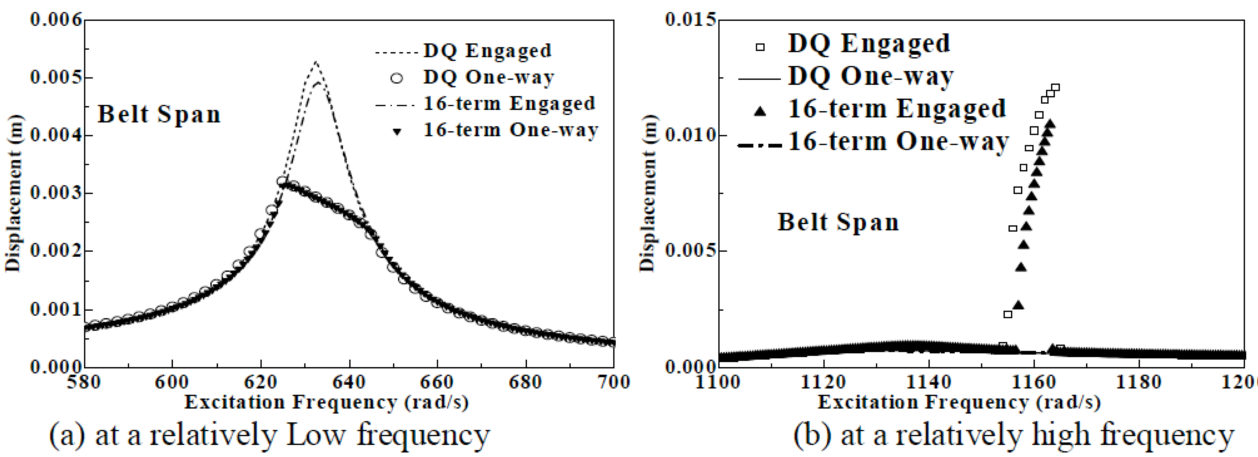


Figure 9: The comparisons of the steady-state response *via* the 16 terms Galerkin truncation and the DQM & IQM.

term, 10-term, 12-term and 16-term, are investigated by comparing with the DQM & IQM. As the comparisons shown in Figures **8** and **9**, the 6-term and the 16-term Galerkin truncation predict the periodic response with qualitative differences. Moreover, the comparisons showed that the study on steady-state responses of the pulley-belt system with a one-way clutch and belt flexural rigidity needs 16-term truncation.

6. CONCLUDING REMARKS

The up-to-date survey of the knowledge on the convergence of the Galerkin method for nonlinear dynamics of the continuous structure in this paper allows for stating that this subject belongs to one of the most currently developed in the last 20 years. The author hopes that this manuscript was able to draw a clear picture of current research on the Galerkin method and the convergence of the Galerkin method for nonlinear dynamics of the continuous structures, especially of the axially moving systems, the continua on the nonlinear foundation, and the belt-pulley systems. One can draw the following conclusions and recommendations for future research work:

- The Galerkin method has been widely used to analyze the nonlinear dynamics of the continuous structure and has been proved as a powerful and efficient tool *via* comparison with other methods, such as the FDM, the ANSYS software, and the multiple time scales method.
- The basis function affects the convergence of the Galerkin method. A set of appropriate basis function improves the convergence of the series expansion.
- The non-trivial equilibrium of the dynamical systems, such as axially moving systems and pulley-belt systems, slows down the convergence of the mode truncation.
- For the future research applications, the convergence of the Galerkin truncation can be

predicted by the natural frequencies of the continuous structure, the property of slower growth in the natural frequency of the structure causes lack of convergence.

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