On Special Strong Differential Superordinations Using Sălăgean and Ruscheweyh Operators

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Abstract: In the present paper we obtain strong differential superordinations for the differential operator L_α^n defined as a convex combination of the extended Sălăgean operator and the extended Ruscheweyh derivative, $L_\alpha^n : \mathbf{A}_\zeta^* \to \mathbf{A}_\zeta^*$, $L_\alpha^n f(z,\zeta) = (1-\alpha)R^n f(z,\zeta) + \alpha S^n f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, and $\mathbf{A}_\zeta^* = \{f \in \mathbf{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_2(\zeta)z^2 + ?,\ z \in U,\ \zeta \in \overline{U}\}$ is the class of normalized analytic functions, $R^n f(z,\zeta)$ the extended Ruscheweyh derivative, $S^n f(z,\zeta)$ the extended Sălăgean operator.

Keywords: strong differential superordination, best subordinant, differential operator.

INTRODUCTION

Let U be the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathbf{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Denote

$$\mathbf{A}_{n\zeta}^* = \{f \in \mathbf{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta)z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},\ \mathbf{A}_{1\zeta}^* = \mathbf{A}_\zeta^*,$$

and

$$\mathbf{H}^*[a,n,\zeta] = \{f \in \mathbf{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ holomorphic functions, $k \ge n$.

We extend the Sălăgean differential operator and Ruscheweyh derivative to the new class of analytic functions $\mathbf{A}_\zeta^*$ introduced in [GIO2].

Definition 1.1. [1] For $f \in \mathbf{A}_\zeta^*$, $m \in \mathbb{N}$, the operator S^m is defined by $S^m : \mathbf{A}_\zeta^* \to \mathbf{A}_\zeta^*$,

$$S^0 f(z,\zeta) = f(z,\zeta),$$
$$S^1 f(z,\zeta) = zf_z'(z,\zeta), \ldots,$$
$$S^{m+1} f(z,\zeta) = z\left(S^m f(z,\zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 1.1. [1] For $f \in \mathbf{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta)z^j$, we have

$$S^m f(z,\zeta) = z + \sum_{j=2}^{\infty} j^m a_j(\zeta)z^j,\ z \in U,\ \zeta \in \overline{U}.$$

Definition 1.2. [1] For $f \in \mathbf{A}_\zeta^*$, $m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathbf{A}_\zeta^* \to \mathbf{A}_\zeta^*$,

$$R^0 f(z,\zeta) = f(z,\zeta),$$
$$R^1 f(z,\zeta) = zf_z'(z,\zeta), \ldots,$$
$$(m+1)R^{m+1} f(z,\zeta) = z\left(R^m f(z,\zeta)\right)_z' + mR^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 1.2. [1] For $f \in \mathbf{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta)z^j$, we have

$$R^m f(z,\zeta) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m a_j(\zeta)z^j,\ z \in U,\ \zeta \in \overline{U}.$$

Using the extended Sălăgean differential operator and the extended Ruscheweyh derivative we have defined a new differential operator as follows

Definition 1.3. [1] Let $\alpha \ge 0$, $m \in \mathbb{N}$,. The operator $L_\alpha^m : \mathbf{A}_\zeta^* \to \mathbf{A}_\zeta^*$, is defined by

$$L_\alpha^m f(z,\zeta) = (1-\alpha)R^m f(z,\zeta) + \alpha S^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Remark 1.3. [1] For $f \in \mathbf{A}_\zeta^*$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta)z^j$, we obtain

$$L_\alpha^m f(z,\zeta) = z + \sum_{j=2}^{\infty}\left(\alpha j^m + (1-\alpha)C_{m+j-1}^m\right)a_j(\zeta)z^j, \quad z \in U,\ \zeta \in \overline{U}.$$

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [2].

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Definition 1.4. [2] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$, such that $H(z,\zeta)=f(w(z),\zeta)$, for all $\zeta\in\overline{U}$. In such a case we write $H(z,\zeta)\prec\prec f(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Remark 1.4. [2] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.4 is equivalent to $H(0,\zeta)=f(0,\zeta)$, for all $\zeta\in\overline{U}$, and $H(U\times\overline{U})\subset f(U\times\overline{U})$.

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.5. We denote by Q^* the set of functions that are analytic and injective on $\overline{U}\times\overline{U}\setminus E(f,\zeta)$, where $E(f,\zeta)=\{y\in\partial U : \lim_{z\to y} f(z,\zeta)=\infty\}$, and are such that $f_z'(y,\zeta)\neq 0$ for $y\in\partial U\times\overline{U}\setminus E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta)=a$ is denoted by $Q^*(a)$.

We have need the following lemmas to study the strong differential superordinations.

Lemma 1.1. *Let* $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma\geq 0$. If $p\in\mathbf{H}^*[a,n,\zeta]\cap Q^*$, $p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta)$ is univalent in $U\times\overline{U}$ and

$$h(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)t^{\frac{\gamma}{n}-1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and is the best subordinant.

Lemma 1.2. Let $q(z,\zeta)$ be a convex function in $U\times\overline{U}$ and let $h(z,\zeta)=q(z,\zeta)+\frac{1}{\gamma}zq_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $\operatorname{Re}\gamma\geq 0$.

If $p\in\mathbf{H}^*[a,n,\zeta]\cap Q^*$, $p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta)$ is univalent in $U\times\overline{U}$ and

$$q(z,\zeta)+\frac{1}{\gamma}zq_z'(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\gamma}zp_z'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)t^{\frac{\gamma}{n}-1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is the best subordinant.

MAIN RESULTS

Theorem 2.1. Consider $h(z,\zeta)$ a convex function in $U\times\overline{U}$, with $h(0,\zeta)=1$, $\alpha\geq 0$, $m\in\mathbb{N}$,, $f(z,\zeta)\in\mathbf{A}_\zeta^*$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)dt$, $z\in U$, $\zeta\in\overline{U}$, $\operatorname{Re}c>-2$, and suppose that $\left(L_\alpha^m f(z,\zeta)\right)_z'$ is univalent in $U\times\overline{U}$, $\left(L_\alpha^m F(z,\zeta)\right)_z'\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$ and

$$h(z,\zeta)\prec\prec\left(L_\alpha^m f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},\tag{2.1}$$

then

$$q(z,\zeta)\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt$. The function q is convex and it is the best subordinant.

Proof. We obtain the relation by differentiating $L_\alpha^m F(z,\zeta)$ with respect to z,

$$(c+1)L_\alpha^m F(z,\zeta)+z\left(L_\alpha^m F(z,\zeta)\right)_z'=(c+2)L_\alpha^m f(z,\zeta),$$
$$z\in U,\ \zeta\in\overline{U},$$

and differentiating again with respect to z, we have

$$\left(L_\alpha^m F(z,\zeta)\right)_z'+\frac{1}{c+2}z\left(L_\alpha^m F(z,\zeta)\right)_{z^2}''=\left(L_\alpha^m f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U}.$$

In this conditions the strong differential superordination (2.1) is

$$h(z,\zeta)\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z'+\frac{1}{c+2}z\left(L_\alpha^m F(z,\zeta)\right)_{z^2}''.$$

Considering

$$p(z,\zeta)=\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime},\ z\in U,\ \zeta\in\overline{U},$$

we obtain the strong differential superordination

$$h(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{c+2}zp_z^{\prime}(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Applying Lemma 1.1. for $n=1$ and $\gamma=c+2$, we obtain

$$q(z,\zeta)\prec\prec p(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U},\ \text{ i.e. } q(z,\zeta)$$
$$\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime},\ \ z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt$. The function q is convex and it is the best subordinant.

Corollary 2.2. For $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, where $\beta\in[0,1)$, $\alpha\geq 0$, $m\in\mathbb{N},$, $f(z,\zeta)\in\mathbf{A}_\zeta^*$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)dt$, $z\in U$, $\zeta\in\overline{U}$, $\operatorname{Re}c>-2$, suppose that $\left(L_\alpha^m f(z,\zeta)\right)_z^{\prime}$ is univalent in $U\times\overline{U}$, $\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime}\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$ and

$$h(z,\zeta)\prec\prec\left(L_\alpha^m f(z,\zeta)\right)_z^{\prime},\ z\in U,\ \zeta\in\overline{U},\tag{2.2}$$

then

$$q(z,\zeta)\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime},\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z)=2\beta-\zeta+\frac{2(c+2)(\zeta-\beta)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. By Theorem 2.1 considering $p(z,\zeta)=\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime}$, we obtain the strong differential superordination (2.2)

$$h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+$$
$$\frac{1}{c+2}zp_z^{\prime}(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Applying Lemma 1.1. for $n=1$ and $\gamma=c+2$, we obtain $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.,

$$q(z,\zeta)=\frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt=\frac{c+2}{z^{c+2}}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}t^{c+1}dt$$
$$=2\beta-\zeta+\frac{2(c+2)(\zeta-\beta)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{t+1}dt\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime},$$
$$z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem 2.3. Consider $q(z,\zeta)$ a convex function in $U\times\overline{U}$, $h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq_z^{\prime}(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, $\operatorname{Re}c>-2$, $\alpha\geq 0$, $m\in\mathbb{N}$, $f\in\mathbf{A}_\zeta^*$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)dt$, $z\in U$, $\zeta\in\overline{U}$. Suppose that $\left(L_\alpha^m f(z,\zeta)\right)_z^{\prime}$ is univalent in $U\times\overline{U}$, $\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime}\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$ and

$$h(z,\zeta)\prec\prec\left(L_\alpha^m f(z,\zeta)\right)_z^{\prime},\ z\in U,\ \zeta\in\overline{U},\tag{2.3}$$

then

$$q(z,\zeta)\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt$. The function q is the best subordinant.

Proof. We obtain the relation by differentiating $L_\alpha^m F(z,\zeta)$ with respect to z,

$$(c+1)L_\alpha^m F(z,\zeta)+z\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime}=$$
$$(c+2)L_\alpha^m f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and differentiating again with respect to z, we have

$$\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime}+\frac{1}{c+2}z\left(L_\alpha^m F(z,\zeta)\right)_{z^2}^{\prime\prime}=\left(L_\alpha^m f(z,\zeta)\right)_z^{\prime},$$
$$z\in U,\ \zeta\in\overline{U}.$$

The strong differential superordination (2.3) becomes

$$h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq_z^{\prime}(z,\zeta)\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime}$$
$$+\frac{1}{c+2}z\left(L_\alpha^m F(z,\zeta)\right)_{z^2}^{\prime\prime}$$
$$m\in\mathbb{N},$$

and considering

$$p(z,\zeta)=\left(L_\alpha^m F(z,\zeta)\right)_z^{\prime},\quad z\in U,\ \zeta\in\overline{U}.$$

we obtain

$$h(z,\zeta)=q(z,\zeta)+\frac{1}{c+2}zq_z'(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{c+2}zp_z'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1.2. for $n=1$ and $\gamma=c+2$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta)\prec\prec\left(L_\alpha^m F(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{c+2}{z^{c+2}}\int_0^z h(t,\zeta)t^{c+1}dt$. The function q is the best subordinant.

Theorem 2.4. Consider $h(z,\zeta)$ convex function, $h(0,\zeta)=1$, $\alpha\ge 0$, $m\in\mathbb{N}$, $f\in\mathbf{A}_\zeta^*$ and suppose that $\left(L_\alpha^m f(z,\zeta)\right)_z'$ is univalent and $\frac{L_\alpha^m f(z,\zeta)}{z}\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$. If

$$h(z,\zeta)\prec\prec\left(L_\alpha^m f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},\tag{2.4}$$

then

$$q(z,\zeta)\prec\prec\frac{L_\alpha^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Proof. Denote $p(z,\zeta)=\frac{L_\alpha^m f(z,\zeta)}{z}=\frac{z+\sum_{j=2}^{\infty}\left(\alpha j^m+(1-\alpha)C_{m+j-1}^m\right)a_j(\zeta)z^j}{z}=1+p_1(\zeta)z+p_2(\zeta)z^2+\ldots,\quad z\in U,$ $\zeta\in\overline{U}$, and obtain that $p\in\mathbf{H}^*[1,1,\zeta]$.

So $L_\alpha^m f(z,\zeta)=zp(z,\zeta)$, $z\in U$ $\zeta\in\overline{U}$ and differentiating with respect to z, we have $\left(L_\alpha^m f(z,\zeta)\right)_z'=p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Then the strong differential superorodination (2.4) becomes

$$h(z,\zeta)\prec\prec p(z,\zeta)+zp_z'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Applying Lemma 1.1. for $n=1$ and $\gamma=1$, we obtain

$$q(z,\zeta)\prec\prec p(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\ \text{i.e. } q(z,\zeta)\prec\prec\frac{L_\alpha^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Corollary 2.5. Consider $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ a convex function in $U\times\overline{U}$, $0\le\beta<1$, $\alpha\ge 0$, $m\in\mathbb{N}$, $f\in\mathbf{A}_\zeta^*$ and suppose that $\left(L_\alpha^m f(z,\zeta)\right)_z'$ is univalent and $\frac{L_\alpha^m f(z,\zeta)}{z}\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$. If

$$h(z,\zeta)\prec\prec\left(L_\alpha^m f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},$$

then

$$q(z,\zeta)\prec\prec\frac{L_\alpha^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(z+1)$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Considering $p(z,\zeta)=\frac{L_\alpha^m f(z,\zeta)}{z}$, the strong differential superordination (2.5) becomes

$$h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp_z'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

From Lemma 1.1. with $n=1$ and $\gamma=1$, we obtain $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt=\frac{1}{z}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}dt$$

$$=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(z+1)\prec\prec\frac{L_\alpha^m f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem 2.6. For $q(z,\zeta)$ convex in $U\times\overline{U}$ define $h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)$. Let $\alpha\ge 0$, $m\in\mathbb{N}$,, $f\in\mathbf{A}_\zeta^*$, and suppose that $\left(L_\alpha^m f(z,\zeta)\right)_z'$ is univalent, $\frac{L_\alpha^m f(z,\zeta)}{z}\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$ and satisfies the strong differential superordination

$$h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)\prec\prec\left(L_\alpha^m f(z,\zeta)\right)_z',\quad z\in U,\ \zeta\in\overline{U},\tag{2.6}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is the best subordinant.

Proof Consider

$$p(z,\zeta) = \frac{L_\alpha^m f(z,\zeta)}{z} = \frac{z + \sum_{j=2}^{\infty}\left(\alpha j^m + (1-\alpha)C_{m+j-1}^m\right)a_j(\zeta)z^j}{z}$$
$$= 1 + \sum_{j=2}^{\infty}\left(\alpha j^m + (1-\alpha)C_{m+j-1}^m\right)a_j(\zeta)z^{j-1} = 1 + \sum_{j=1}^{\infty}p_j(\zeta)z^j, \quad z \in U, \zeta \in \overline{U}.$$

Differentiating with respect to z we obtain $\left(L_\alpha^m f(z,\zeta)\right)_z' = p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U, \quad \zeta \in \overline{U}$, and the strong difefrential superoordination (2.6) becomes

$$q(z,\zeta) + zq_z'(z,\zeta) \quad p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Applying Lemma 1.2. for $n=1$ and $\gamma=1$, we obtain

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.}$$
$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt \prec\prec \frac{L_\alpha^m f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant.

Theorem 2.7. Let $h(z,\zeta)$ a convex function, $h(0,\zeta)=1$, $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathbf{A}_\zeta^*$ and suppose that $\left(\frac{zL_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{L_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)} \in \mathbf{H}^*[1,1,\zeta] \cap Q^*$. If

$$h(z,\zeta) \prec\prec \left(\frac{zL_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{2.7}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Proof. Denote

$$p(z,\zeta) = \frac{L_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)} =$$
$$\frac{z + \sum_{j=2}^{\infty}\left(\alpha j^{m+1} + (1-\alpha)C_{m+j}^{m+1}\right)a_j(\zeta)z^j}{z + \sum_{j=2}^{\infty}\left(\alpha j^m + (1-\alpha)C_{m+j-1}^m\right)a_j(\zeta)z^j}.$$

Differentiating with respect to z we obtain $p_z'(z,\zeta) = \frac{\left(L_\alpha^{m+1}f(z,\zeta)\right)_z'}{L_\alpha^m f(z,\zeta)} - p(z,\zeta)\cdot\frac{\left(L_\alpha^m f(z,\zeta)\right)_z'}{L_\alpha^m f(z,\zeta)}$ and $p(z,\zeta) + z\cdot p_z'(z,\zeta) = \left(\frac{zL_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z'$.

The strong differential superorodination (2.7) is

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}$$

and from Lemma 1.1., where $n=1$ and $\gamma=1$, we obtain

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e. } q(z,\zeta)$$
$$\prec \frac{L_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is convex and it is the best subordinant.

Corollary 2.8. For $h(z,\zeta) = \frac{\zeta + (2\beta-\zeta)z}{1+z}$ a convex function in $U \times \overline{U}$, $0 \leq \beta < 1$, $\alpha \geq 0$, $m \in \mathbb{N}$, $f \in \mathbf{A}_\zeta^*$, we suppose that $\left(\frac{zL_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{L_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)} \in \mathbf{H}^*[1,1,\zeta] \cap Q^*$. If

$$h(z,\zeta) \prec\prec \left(\frac{zL_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{2.8}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(z+1)$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant.

Proof. By Theorem 2.7. for $p(z,\zeta) = \frac{L_\alpha^{m+1}f(z,\zeta)}{L_\alpha^m f(z,\zeta)}$, the strong differential superordination (2.8) becomes

$$h(z,\zeta) = \frac{\zeta + (2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

and we have $q(z,\zeta) \prec\prec p(z,\zeta)$, where

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta) = \frac{1}{z}\int_0^z \frac{\zeta + (2\beta-\zeta)t}{1+t}dt$$

$$= 2\beta - \zeta + \frac{2(\zeta - \beta)}{z}\ln(z+1) \prec\prec \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem 2.9. Consider $q(z,\zeta)$ a convex function and $h(z,\zeta) = q(z,\zeta) + zq_z'(z,\zeta)$. Let $\alpha \ge 0$, $m \in \mathbb{N}$, $f \in \mathbf{A}_\zeta^*$ and suppose that $\left(\frac{zL_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z'$ is univalent and $\frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)} \in \mathbf{H}^*[1,1,\zeta] \cap Q^*$. If

$$h(z,\zeta) = q(z,\zeta) + zq_z'(z,\zeta) \prec\prec \left(\frac{zL_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{2.9}$$

then

$$q(z,\zeta) \prec\prec \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt$. The function q is the best subordinant.

Proof. Denote

$$p(z,\zeta) = \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)} = \frac{z + \sum_{j=2}^\infty \left(\alpha j^{m+1} + (1-\alpha) C_{m+j}^{m+1}\right) a_j(\zeta) z^j}{z + \sum_{j=2}^\infty \left(\alpha j^m + (1-\alpha) C_{m+j-1}^{m}\right) a_j(\zeta) z^j}.$$

Differentiating with respect to z we obtain $p_z'(z,\zeta) = \frac{\left(L_\alpha^{m+1} f(z,\zeta)\right)_z'}{L_\alpha^m f(z,\zeta)} - p(z,\zeta) \cdot \frac{\left(L_\alpha^m f(z,\zeta)\right)_z'}{L_\alpha^m f(z,\zeta)}$ and $p(z,\zeta) + z \cdot p_z'(z,\zeta) = \left(\frac{zL_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}\right)_z'$.

The strong differential superorodination (2.9) has the form

$$h(z,\zeta) = q(z,\zeta) + zq_z'(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

and applying Lemma 1.2. for $n = 1$ and $\gamma = 1$, we obtain

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \text{ i.e.}$$

$$q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)dt \prec\prec \frac{L_\alpha^{m+1} f(z,\zeta)}{L_\alpha^m f(z,\zeta)}, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant.

Theorem 2.10. Consider $h(z,\zeta)$ a convex function in $U \times \overline{U}$, with $h(0,\zeta) = 1$, and $\alpha \ge 0$, $m \in \mathbb{N}$, $f \in \mathbf{A}_\zeta^*$. Suppose $\left(L_\alpha^{m+1} f(z,\zeta)\right)_z' + \frac{(1-\alpha)mz(R^m f(z,\zeta))_{z^2}''}{m+1}$ is univalent and $[L_\alpha^m f(z,\zeta)]_z' \in \mathbf{H}^*[1,1,\zeta] \cap Q^*$. If

$$h(z,\zeta) \prec\prec \left(L_\alpha^{m+1} f(z,\zeta)\right)_z' + \frac{(1-\alpha)mz(R^m f(z,\zeta))_{z^2}''}{m+1}, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.10}$$

holds, then

$$q(z,\zeta) \prec\prec [L_\alpha^m f(z,\zeta)]_z', \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)$. The function is convex and it is the best subordinant.

Proof. The strong differential superordination (2.10) becomes

$$h(z,\zeta) \prec\prec \left((1-\alpha)R^{m+1} f(z,\zeta) + \alpha S^{m+1} f(z,\zeta)\right)_z' + \frac{(1-\alpha)mz\left(R^m f(z,\zeta)\right)_{z^2}''}{m+1},$$

$$z \in U,\ \zeta \in \overline{U}.$$

Denote

$$p(z,\zeta) = (1-\alpha)\left(R^m f(z,\zeta)\right)_z' + \alpha\left(S^m f(z,\zeta)\right)_z' = \left(L_\alpha^m f(z,\zeta)\right)_z' \tag{2.11}$$

$$= 1 + \sum_{j=2}^\infty \left(\alpha j^{m+1} + (1-\alpha) jC_{m+j-1}^m\right) a_j(\zeta) z^{j-1} = 1 + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \ldots.$$

Using the notation in (2.11), the strong differential superordination becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta),$$

and applyinh Lemma 1.1. for $n = 1$ and $\gamma = 1$, we obtain

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \text{ i.e. } q(z,\zeta) \prec\prec \left(L_\alpha^m f(z,\zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{z}\int_0^z h(t,\zeta)$. The function q is convex and it is the best subordinant.

Corollary 2.11. For $h(z)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ a convex function in $U\times\overline{U}$, $0\le\beta<1$, $\alpha\ge 0$, $m\in\mathbb{N}$, $f\in \mathbf{A}^*_\zeta$, suppose that $\left(L^{m+1}_\alpha f(z,\zeta)\right)'_z+\frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}$ is univalent in $U\times\overline{U}$ and $[L^m_\alpha f(z,\zeta)]'_z\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$. If

$$h(z,\zeta)\prec\prec[L^{m+1}_\alpha f(z,\zeta)]'_z+\frac{(1-\alpha)mz\left(R^m f(z,\zeta)\right)''_{z^2}}{m+1},$$

$z\in U$, $\zeta\in\overline{U}$,

then

$$q(z,\zeta)\prec\prec\left(L^m_\alpha f(z,\zeta)\right)'_z,\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(z+1)$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Theorem 2.10. for $p(z,\zeta)=\left(L^m_\alpha f(z,\zeta)\right)'_z$, give $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)=\frac{1}{z}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}dt$$

$$=2\beta-\zeta+\frac{2(\zeta-\beta)}{z}\ln(z+1)\prec\prec\left(L^m_\alpha f(z,\zeta)\right)'_z,\quad z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem 2.12. Consider $q(z,\zeta)$ a convex function in $U\times\overline{U}$ and $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$, $\alpha\ge 0$, $m\in\mathbb{N}$, $f\in\mathbf{A}^*_\zeta$. Suppose $\left(L^{m+1}_\alpha f(z,\zeta)\right)'_z+\frac{(1-\alpha)mz(R^m f(z,\zeta))''_{z^2}}{m+1}$ is univalent in $U\times\overline{U}$ and $[L^m_\alpha f(z,\zeta)]'_z\in\mathbf{H}^*[1,1,\zeta]\cap Q^*$. If

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec[L^{m+1}_\alpha f(z,\zeta)]'_z+\frac{(1-\alpha)mz\left(R^m f(z,\zeta)\right)''_{z^2}}{m+1},$$

$z\in U$, $\zeta\in\overline{U}$, then

$$q(z,\zeta)\prec\prec\left(L^m_\alpha f(z,\zeta)\right)'_z,\ z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)$. The function q is te best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.10 for $p(z,\zeta)=\left(L^m_\alpha f(z,\zeta)\right)'_z$, the strong differential superordination (2.13) becomes

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta),\ z\in U,\ \zeta\in\overline{U},$$

and from Lemma 1.2. for $n=1$ and $\gamma=1$, we obtain $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)\prec\prec\left(L^m_\alpha f(z,\zeta)\right)'_z,\quad z\in U,\ \zeta\in\overline{U}.$$

The function q is the best subordinant.

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