On Traces of n-Additive Mappings on Semiprime Ring

Asma Ali, Najat Muthana* and Kapil Kumar

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia

Abstract: Let R be a ring with centre $Z(R)$. In this paper we prove that a nonzero Lie ideal L of a semiprime ring R of characteristic different from $(2^n - 2)$ is central if it satisfies one of the following:

$(i)\ f(xy) \mp [x,y] \in Z(R)$, $(ii)\ f(xy) \mp [y,x] \in Z(R)$, $(iii)\ f(xy) \mp xy \in Z(R)$, $(iv)\ f(xy) \mp yx \in Z(R)$, $(v)\ f([x,y]) \mp [x,y] \in Z(R)$, $(vi)\ f([x,y]) \mp [y,x] \in Z(R)$, $(vii)\ f([x,y]) \mp xy \in Z(R)$, $(viii)\ f([x,y]) \mp yx \in Z(R)$, $(ix)\ f(xy) \mp f(x) \mp [x,y] \in Z(R)$, $(x)\ f(xy) \mp f(y) \mp [x,y] \in Z(R)$, $(xi)\ f([x,y]) \mp f(x) \mp [y,x] \in Z(R)$, $(xii)\ f([x,y]) \mp f(x) \mp [y,x] \in Z(R)$, $(xiii)\ f([x,y]) \mp f(y) \mp [x,y] \in Z(R)$, $(xiv)\ f([x,y]) \mp f(y) \mp [y,x] \in Z(R)$, $(xv)\ f([x,y]) \mp f(xy) \mp [x,y] \in Z(R)$, $(xvi)\ f([x,y]) \mp f(xy) \mp [y,x] \in Z(R)$, $(xvii)\ f(x)f(y) \mp [x,y] \in Z(R)$, $(xviii)\ f(x)f(y) \mp [y,x] \in Z(R)$, $(xix)\ f(x)f(y) \mp xy \in Z(R)$, $(xx)\ f(x)f(y) \mp yx \in Z(R)$ for all $x,y \in L$, where f stands for the trace of an n-additive map $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$

Keywords: Semiprime rings, Lie ideals, n-additive maps, Trace of n-additive maps.

INTRODUCTION

Throughout the paper, R will denote an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = \{0\}$ implies that either $a = 0$ or b=0 (resp. $aRa = \{0\}$ implies that $a = 0$). For each pair of elements $x, y \in R$ we shall write $[x,y]$ the commutator $xy - yx$.

An additive subgroup L of a ring R is said to be a Lie ideal of R if $[U,R] \subseteq U$. An additive map $d: R \to R$ is said to be a derivation if,

$$d(xy) = d(x)y + xd(y)\ for\ all\ x,y \in R$$

An additive map $G: R \to R$ is said to be a generalized derivation if there exists a derivation $d: R \to R$ such that,

$$G(xy) = G(x)y + xd(y)\ for\ all\ x,y \in R$$

Let $n \geq 2$ be a fixed positive integer. A map $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$ is said to be symmetric (permuting) if,

$F(x_{(1)}, x_{(2)}, \ldots, x_{(n)}) = F(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for all $x_i \in R$ for every permutation $\{(\pi(1), \pi(2), \ldots, \pi(n))\}$. The map $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$ is said to be n-additive if $F(x_{(1)}, x_{(2)}, \ldots, x_{(n)})$ is additive in each variable $x_{(i)}$; $i = 1,2,\ldots,n$ that is,

$F(x_{(1)}, x_{(2)}, \ldots, x_{(i)}\{+y_{(i)}, \ldots, x_{(n)})$= $F(x_{(1)}, x_{(2)}, \ldots, x_{(i)}, \ldots, x_{(n)})$+$F(x_{(1)}, x_{(2)}, \ldots, y_{(i)}, \ldots, x_{(n)})$ for all $x_i, \sim y_i \in R$ and $i = 1,2,\ldots,n$.

The mapping $f: R \to R$ defined by f(x) = F(x,x,...,x) is called the trace of F. It is obvious that in case when $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$ is a symmetric n-additive map, the trace f of F satisfies the relation $f(x+y) = f(x) + f(y) + \sum_{k=1}^{n-1}\binom{n}{k} h_k(x,y)$.

In 1992 Daif and Bell [3, Theorem 1] proved that if a semiprime ring R admits a derivation d such that $d([x,y]) - [x,y] \in Z(R)\ for\ all\ x,y \in R$, then R is commutative. Further first author [1] investigated the commutativity of a prime ring R admitting a generalized derivation G satisfying one of the following:$(i)\ G(xy) \mp xy \in Z(R)$, $(ii)\ G(xy) \mp yx \in Z(R)$, $(iii)\ G(x)G(y) \mp xy \in Z(R)\ for\ all\ x,y$ in some appropriate subset of R.

Motivated by the aforementioned results are investigate the following conditions:

$(i)\ f(xy) \mp [x,y] \in Z(R)$,

$(ii)\ f(xy) \mp [y,x] \in Z(R)$,

$(iii)\ f(xy) \mp xy \in Z(R)$,

$(iv)\ f(xy) \mp yx \in Z(R)$,

$(v)\ f([x,y]) \mp [x,y] \in Z(R)$,

$(vi)\ f([x,y]) \mp [y,x] \in Z(R)$,

$(vii)\ f([x,y]) \mp xy \in Z(R)$,

$(viii)\ f([x,y]) \mp yx \in Z(R)$,

$(ix)\ f(xy) \mp f(x) \mp [x,y] \in Z(R)$,

$(x)\ f(xy) \mp f(y) \mp [x,y] \in Z(R)$,

$(xi)\ f([x,y]) \mp f(x) \mp [y,x] \in Z(R)$,

$(xii)\ f([x,y]) \mp f(x) \mp [y,x] \in Z(R)$,

$(xiii)\ f([x,y]) \mp f(y) \mp [x,y] \in Z(R)$,

*Adress correspondence to this author at the Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia; E-mails: nmuthana@kau.edu.sa; asma_ali2@rediffmail.com; 01kapilmathsamu@gmail.com

E-ISSN: 2409-5761/17

$(xiv)\ f([x,y]) \mp f(y) \mp [y,x] \in Z(R)$,

$(xv)\ f([x,y]) \mp f(xy) \mp [x,y] \in Z(R)$,

$(xvi)\ f([x,y]) \mp f(xy) \mp [y,x] \in Z(R)$,

$(xvii)\ f(x)f(y) \mp [x,y] \in Z(R)$,

$(xviii)\ f(x)f(y) \mp [y,x] \in Z(R)$,

$(xix)\ f(x)f(y) \mp xy \in Z(R)$,

$(xx)\ f(x)f(y) \mp yx \in Z(R)\ for\ all\ x,y \in L$,

a nonzero Lie ideal of R and f trace of n-additive map $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\rightarrow R$.

PRELIMINARY RESULTS

The following Lemmas are essential to prove our Theorems.

Lemma 2.1. [4, Lemma 1] Let R be a semiprime ring and L be a nonzero Lie ideal of R. If $[L,L] \subseteq Z(R)$, then $L \subseteq Z(R)$.

Lemma 2.2. Let R be a semiprime ring and L be a nonzero Lie ideal of R. If $L^2 \subseteq Z(R)$, then $L \subseteq Z(R)$.

Proof. Since $xy \in Z(R)$ for all $x,y \in L$, $xy - yx = [x,y] \in Z(R)\ for\ all\ x,y \in L$. Using Lemma 2.1 we get the required result.

MAIN RESULT

Theorem 3.1. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\rightarrow R$ be a symmetric n-additive mapping and f be the trace of F. If $f(xy) \mp [x,y] \in Z(R)\ x,y \in L$, then $L \subseteq Z(R)$.

Proof. Suppose,

$$f(xy) - [x,y] \in Z(R)\ for\ all\ x,y \in L\ . \quad (3.1)$$

Replacing y by $y+z$ in (3.1), we get

$f(xy) + f(xz) + \sum_{k=1}^{n-1}\binom{n}{k} h_k\,(xy,xz) - [x,y] - [x,z] \in Z(R)$ for all $x,y,z \in L$

From (3.1), we get,

$$\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xz) \in Z(R)\ for\ all\ x,y,z \in L. \quad (3.2)$$

Replacing z by y in (3.2), we find

$\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xz) \in Z(R)$ for all $x,y,z \in L$

that is,

$\binom{n}{1}h_1(xy,xy)+\binom{n}{2}h_2(xy,xy)+\binom{n}{3}h_3(xy,xy) + \ldots + \binom{n}{n-1}h_{n-1}(xy,xy) \in Z(R)$.

This implies,

$$\begin{aligned}&\binom{n}{1}F:(\underbrace{xy,xy,\cdots,xy}_{(n-1)\text{-}times}\rightarrow \underbrace{xy}_{1\text{-}times})\\ &+\binom{n}{2}F(\underbrace{xy,xy,\cdots,xy}_{(n-2)\text{-}times}\rightarrow \underbrace{xy}_{2\text{-}times})\\ &+\binom{n}{3}F(\underbrace{xy,xy,\cdots,xy}_{(n-3)\text{-}times}\rightarrow \underbrace{xy}_{3\text{-}times})\\ &+\cdots+\binom{n}{n-1}F(\underbrace{xy,xy,\cdots,xy}_{(n-1)\text{-}times}\rightarrow \underbrace{xy}_{1\text{-}times})\in Z(R)\end{aligned}$$

for all $x,y \in L$.

This can be written as,

$(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\ldots+\binom{n}{n-1})F(xy,xy,\ldots,xy) \in Z(R)$ for all $x,y \in L$.

This gives,

$$(2^n-2)F(xy,xy,\ldots,xy) \in Z(R)\ for\ all\ x,y \in L.$$

Since R is not of characteristic (2^n-2). Then $F(xy,xy,\ldots,xy) \in Z(R)$ for all $x,y \in L$. This implies that $f(xy) \in Z(R)$. From (3.1), we get $[x,y] \in Z(R)$ for all $x,y \in L$.

This implies that $[L,L] \subseteq Z(R)$. Hence $L \subseteq Z(R)$ by Lemma 2.1.

Similarly, we can prove the result for the case $f(xy) + [x,y] \in Z(R)$ for all $x,y \in L$.

Theorem 3.2. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\rightarrow R$ be a symmetric n-additive mapping and f be the traceof F.

If $f(xy) \mp [y,x] \in Z(R)\ for\ all\ x,y \in L$,

then $L \subseteq Z(R)$.

Proof. The proof runs on the same parallel lines as of Theorem 3.1.

Theorem 3.3. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\rightarrow R$ be a symmetric n-additive mapping and f be the trace of F.

If $f(xy) \mp xy \in Z(R)\ for\ all\ x,y \in L$,

then $L \subseteq Z(R)$.

Proof. Suppose,

$f(xy) - xy \in Z(R)\, for\ all\ x, y \in L$. (3.3)

Replacing y by $y+z$ in (3.3), we get,

$f(xy) + f(xz) + \sum_{k=1}^{n-1}\binom{n}{k} h_k(xy, xz) - xy - xz \in Z(R)\, for\ all\ x, y, z \in L$. (3.4)

Using (3.3), we obtain,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy, xz) \in Z(R)\, for\ all\ x, y, z \in L$. (3.5)

Substituting y for z in (3.5), we get,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy, xy) \in Z(R)\, for\ all\ x, y \in L$. (3.6)

This can be written as,

$\binom{n}{1}h_1(xy, xy) + \binom{n}{2}h_2(xy, xy) + \binom{n}{3}h_3(xy, xy) + ... + \binom{n}{n-1}h_{n-1}(xy, xy) \in Z(R)$. (3.7)

that is,

$$\binom{n}{1}F(\underbrace{xy, xy, \cdots, xy}_{(n-1)\text{-times}}, \underbrace{xy}_{1\text{-times}}) + \binom{n}{2}F(\underbrace{xy, xy, \cdots, xy}_{(n-2)\text{-times}}, \underbrace{xy}_{2\text{-times}}) + \binom{n}{3}F(\underbrace{xy, xy, \cdots, xy}_{(n-3)\text{-times}}, \underbrace{xy}_{3\text{-times}}) + \cdots + \binom{n}{n-1}F(\underbrace{xy}_{1\text{-times}}, \underbrace{xy, xy, \cdots, xy}_{(n-1)\text{-times}}) \in Z(R)$$

for all $x, y \in L$.

This implies,

$(\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{n-1})\ F(xy, xy, \ldots, xy) \in Z(R)\, for\ all\ x, y \in L$.

This gives,

$(2^n - 2)F(xy, xy, \ldots, xy) \in Z(R)\, for\ all\ x, y \in L$.

Since R is not of characteristic $(2^n - 2)$. Then $F(xy, xy, \ldots, xy) \in Z(R)\, for\ all\ x, y \in L$. This implies that $f(xy) \in Z(R)$. Using (3.3), we have $xy \in Z(R)$ for all $x, y \in L$.

Hence $L^2 \subseteq Z(R)$ and by Lemma 2.2, $L \subseteq Z(R)$.

Similarly, we can prove the result if $f(xy) + xy \in Z(R)$ for all $x, y \in L$.

Theorem 3.4. Let R be a semiprime ring of characteristic not $(2^n - 2)$ and L be a nonzero Lie ideal of R. Let $F: \underbrace{R \times R \times \cdots \times R}_{n\text{-times}} \to R$ be a symmetric n-additive mapping and f be the trace of F. If $f(xy) \mp yx \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof. The proof runs on the same parallel lines as of Theorem 3.3.

Theorem 3.5. Let R be a semiprime ring of characteristic not $(2^n - 2)$ and L be a nonzero Lie ideal of R. Let $F: \underbrace{R \times R \times \cdots \times R}_{n\text{-times}} \to R$ be a symmetric n-additive mapping and f be the trace of F. If $f([x, y]) \mp [x, y] \in Z(R)$ for all $x, y \in L$, then $L \subseteq Z(R)$.

Proof. Let

$f([x, y]) - [x, y] \in Z(R)\, for\ all\ x, y \in L$. (3.8)

Replacing y by $y+z$ in (3.8), we have

$$f([x, y] + [x, z]) - [x, y] - [x, z] \in Z(R)$$

for all $x, y, z \in L$.

This implies,

$f([x, y]) + f([x, z]) + \sum_{k=1}^{n-1}\binom{n}{k} h_k([x,y],[x,z]) - [x,y] - [x,z] \in Z(R)$.

Using (3.8), we get,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k([x, y], [x, z]) \in Z(R)$ for all $x, y, z \in L$.

Substituting y for z in (3.9), we obtain,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k([x, y], [x, y]) \in Z(R)$ for all $x, y, z \in L$.

that is,

$\binom{n}{1}h_1([x,y],[x,y]) + \binom{n}{2}h_2([x,y],[x,y]) + \binom{n}{3}h_3([x,y],[x,y]) + ... + \binom{n}{n-1}h_{n-1}([x,y],[x,y]) \in Z(R)$.

This gives,

$$\binom{n}{1}F(\underbrace{[x,y],[x,y],\cdots,[x,y]}_{(n-1)\text{-times}}, \underbrace{[x,y]}_{1\text{-times}}) + \binom{n}{2}F(\underbrace{[x,y],[x,y],\cdots,[x,y]}_{(n-2)\text{-times}}, \underbrace{[x,y]}_{2\text{-times}}) + \binom{n}{3}F(\underbrace{[x,y],[x,y],\cdots,[x,y]}_{(n-3)\text{-times}}, \underbrace{[x,y]}_{3\text{-times}}) + \cdots + \binom{n}{n-1}F(\underbrace{[x,y]}_{1\text{-times}}, \underbrace{[x,y],[x,y],\cdots,[x,y]}_{(n-1)\text{-times}}) \in Z(R)$$

Therefore,

$(\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{n-1})\, F([x, y], [x, y], \ldots, [x, y]) \in Z(R)$ for all $x, y \in L$.

This implies,

$(2^n-2)F([x,y],[x,y],\dots,[x,y]) \in Z(R)$ for all $x,y \in L$.

Since R is not of characteristic (2^n-2), we find $F([x,y],[x,y],\dots,[x,y]) \in Z(R)$ for all $x,y \in L$. This implies that

$$f([x,y]) \in Z(R).$$

In view of (3.8), (3.10) yields that $[x,y] \in Z(R)$ for all $x,y \in L$. Thus we get $[L,L] \subseteq Z(R)$ and by Lemma 2.1, $L \subseteq Z(R)$.

Similarly, we can prove that the result if $f([x,y]) + [x,y] \in Z(R)$ for all $x,y \in L$.

Using similar arguments as we have done in the proof of the Theorem 3.5, we can prove the following:

Theorem 3.6. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\dots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F.

If $f([x,y]) \mp [y,x] \in Z(R)\ for\ all\ x,y \in L$,

then $L \subseteq Z(R)$.

Theorem 3.7 Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\dots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F.

If $f([x,y]) \mp xy \in Z(R)\ for\ all\ x,y \in L$,

then $L \subseteq Z(R)$.

Proof. Let

$f([x,y]) - xy \in Z(R)\ for\ all\ x,y \in L$. (3.11)

Replacing y by $y+z$ in (3.8), we have,

$f([x,y]+[x,z]) - xy - xz \in Z(R)\ for\ all\ x,y,z \in L$.

This implies,

$f([x,y]) + f([x,z]) + \sum_{k=1}^{n-1}\binom{n}{k} h_k$ ([x,y],[x,z])-xy-xz $\in Z(R)$.(3.12)

Using (3.11), we obtain,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k([x,y],[x,z]) \in Z(R)\ for\ all\ x,y,z \in L$.

This can be written as,

$\binom{n}{1}h_1([x,y],[x,z])+\binom{n}{2}h_2([x,y],[x,z])$

$+\binom{n}{3}h_3([x,y],[x,z])+\dots$

$+\binom{n}{n-1}h_{n-1}([x,y],[x,z]) \in Z(R)$. (3.13)

Substituting y for z in (3.13), we obtain

$\binom{n}{1}h_1([x,y],[x,y])+\binom{n}{2}h_2([x,y],[x,y])$

$+\binom{n}{3}h_3([x,y],[x,y])+\dots+\binom{n}{n-1}h_{n-1}([x,y],[x,y]) \in Z(R)$.

This gives,

$$\binom{n}{1}F(\underbrace{[x,y],[x,y],\dots,[x,y]}_{(n-1)\text{-times}},\ \underbrace{[x,y])}_{1\text{-times}}$$
$$+\binom{n}{2}F(\underbrace{[x,y],[x,y],\dots,[x,y]}_{(n-2)\text{-times}},\ \underbrace{[x,y])}_{2\text{-times}}$$
$$+\binom{n}{3}F(\underbrace{[x,y],[x,y],\dots,[x,y]}_{(n-3)\text{-times}},\ \underbrace{[x,y])}_{3\text{-times}}$$
$$+\dots+\binom{n}{n-1}F(\underbrace{[x,y],}_{1\text{-times}}\ \underbrace{[x,y],[x,y],\dots,[x,y])}_{(n-1)\text{-times}} \in Z(R)$$

that is,

$(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\dots+\binom{n}{n-1})$

$F([x,y],[x,y],\dots,[x,y]) \in Z(R)\ for\ all\ x,y \in L$.

Thus,

$(2^n-2)F([x,y],[x,y],\dots,[x,y]) \in Z(R)\ for\ all\ x,y \in L$. (3.14)

Since R is not of characteristic (2^n-2), (3.14) yields that,

$F([x,y],[x,y],\dots,[x,y]) \in Z(R)\ for\ all\ x,y \in L$,

then we have $f([x,y]) \in Z(R)\ for\ all\ x,y \in L$. Using (3.11), we get xy $\in Z(R)\ for\ all\ x,y \in L$.

Thus $L^2 \subseteq Z(R)$ and application of Lemma 2.2, we get the result.

Similarly, we can prove that the result if $f([x,y]) + xy \in Z(R)\ for\ all\ x,y \in L$.

Theorem 3.8.

Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\dots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f([x,y]) \mp yx \in Z(R)\ for\ all\ x,y \in L$, then $L \subseteq Z(R)$.

Proof. The proof runs on the same parallel lines as that of Theorem **3.7**.

Theorem 3.9.

Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$ be a symmetric n-additive mapping and f be the trace of F.

If $f(xy) \mp f(x) \mp [x,y] \in Z(R)$ $for\ all\ x,y \in L$,

then $L \subseteq Z(R)$.

Proof. Suppose,

$f(xy) - f(x) - [x,y] \in Z(R)$ $for\ all\ x,y \in L$. (3.15)

Replacing x by $x+z$ in (3.15), we have

$$f(xy+zy) - f(x+z) - [x,y] - [z,y] \in Z(R)\ for\ all\ x,y,z \in L.$$

This can be written as,

$f(xy) + f(zy) + \sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,zy) - f(x)$

$-f(z) - \sum_{k=1}^{n-1}\binom{n}{k} h_k(x,z) - [x,y] - [z,y] \in Z(R)$.

This implies,

$$f(xy) - f(x) - [x,y] + f(zy) - f(z) - [z,y]$$

$+\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,zy) - \sum_{k=1}^{n-1}\binom{n}{k} h_k(x,z) \in Z(R)$.

Using (3.16), we get

$\sum_{k=1}^{n-1}\binom{n}{k}$ $h_k(xy,zy)$- $\sum_{k=1}^{n-1}\binom{n}{k}$ $h_k(x,z)$ $\in Z(R)$ for all $x,y,z \in L$. (3.16)

Substituting x for z in (3.17), we obtain

$\binom{n}{k}h_1(xy,xy)+\binom{n}{k}h_2(xy,xy)+\binom{n}{k}h_3(xy,xy)+\ldots$

$+\binom{n}{k}h_{n-1}(xy,xy)-\binom{n}{k}h_1(x,x)$

$-\binom{n}{k}h_2(x,x)$ $-\binom{n}{k}h_3(x,x)-\ldots-\binom{n}{k}h_{n-1}(x,x) \in Z(R)$

that is,

$$\Bigg(\binom{n}{1}F(\underbrace{xy,xy,\ldots,xy}_{(n-1)\text{-}times},\underbrace{xy}_{1\text{-}times}) + \binom{n}{2}F(\underbrace{xy,xy,\ldots,xy}_{(n-2)\text{-}times},\underbrace{xy}_{2\text{-}times}) + \binom{n}{3}F(\underbrace{xy,xy,\ldots,xy}_{(n-3)\text{-}times},\underbrace{xy}_{3\text{-}times}) + \cdots + \binom{n}{n-1}F(\underbrace{xy}_{1\text{-}times},\underbrace{xy,xy,\ldots,xy}_{(n-1)\text{-}times})$$

$$-\binom{n}{1}F(\underbrace{x,x,\ldots,x}_{(n-1)\text{-}times},\underbrace{x}_{1\text{-}times}) - \binom{n}{2}F(\underbrace{x,x,\ldots,x}_{(n-2)\text{-}times},\underbrace{x}_{2\text{-}times}) - \binom{n}{3}F(\underbrace{x,x,\ldots,x}_{(n-3)\text{-}times},\underbrace{x}_{3\text{-}times}) - \cdots - \binom{n}{n-1}F(\underbrace{x}_{1\text{-}times},\underbrace{x,x,\ldots,x}_{(n-1)\text{-}times})\Bigg) \in Z(R).$$

Therefore,

$(2^n-2)\big(F(xy,xy,\ldots xy) - F(x,x,\ldots,x)\big) \in Z(R)$ $x,y \in L$. (3.17)

Since R is not of characteristic (2^n-2), (3.18) yields that $\big(F(xy,xy,\ldots xy) - F(x,x,\ldots,x)\big) \in Z(R)$, we have $f(xy) - f(x) \in Z(R)$ $for\ all\ x,y \in L$. Using (3.16), we get $[x,y] \in Z(R)$ $for\ all\ x,y \in L$. Thus we get $[L,L] \subseteq Z(R)$ and by Lemma 2.1, $L \subseteq Z(R)$.

Similarly, we can prove the result if $f(xy) + f(x) + [x,y] \in Z(R)$ $for\ all\ x,y \in L$.

Theorem 3.10.

Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f(xy) \mp f(y) \mp [x,y] \in Z(R)$ $for\ all\ x,y \in L$, then $L \subseteq Z(R)$.

Proof. Let,

$f(xy) - f(y) - [x,y] \in Z(R)$ $for\ all\ x,y \in L$. (3.18)

Replacing y by $y+z$ in (3.19), we have

$f(xy) + f(xz) + \sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xz) - f(y) - f(z)$

$-\sum_{k=1}^{n-1}\binom{n}{k} h_k(y,z) - [x,y] - [x,z] \in Z(R)$.

Using (3.18), we get,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xz) - \sum_{k=1}^{n-1}\binom{n}{k} h_k(y,z) \in Z(R)$ for all $x,y,z \in L$. (3.19)

Substituting y for z in (3.19), we obtain,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xy) - \sum_{k=1}^{n-1}\binom{n}{k} h_k(y,y) \in Z(R)$

for all $x,y, \in L$.

This can be written as,

$\binom{n}{1}h_1(xy,xy)+\binom{n}{2}h_2(xy,xy)+\binom{n}{3}h_3(xy,xy)+\ldots+\binom{n}{n-1}h_{n-1}(xy,xy)$

$-\binom{n}{1}h_1(y,y)-\binom{n}{2}h_2(y,y)-\binom{n}{3}h_3(y,y)-\cdots$

$-\binom{n}{n-1}h_{n-1}(y,y)\in Z(R)$

that is,

$$\left(\binom{n}{1}F(\underbrace{xy,xy,\ldots,xy}_{(n-1)\text{-times}},\underbrace{xy}_{1\text{-times}})\right.$$
$$+\binom{n}{2}F(\underbrace{xy,xy,\ldots,xy}_{(n-2)\text{-times}},\underbrace{xy}_{2\text{-times}})$$
$$+\binom{n}{3}F(\underbrace{xy,xy,\ldots,xy}_{(n-3)\text{-times}},\underbrace{xy}_{3\text{-times}})$$
$$+\ldots+\binom{n}{n-1}F(\underbrace{xy}_{1\text{-times}},\underbrace{xy,xy,\ldots,xy}_{(n-1)\text{-times}})$$
$$-\binom{n}{1}F(\underbrace{y,y,\ldots,y}_{(n-1)\text{-times}},\underbrace{y}_{1\text{-times}})$$
$$-\binom{n}{2}F(\underbrace{y,y,\ldots,y}_{(n-2)\text{-times}},\underbrace{y}_{2\text{-times}})$$
$$-\binom{n}{3}F(\underbrace{y,y,\ldots,y}_{(n-3)\text{-times}},\underbrace{y}_{3\text{-times}})$$
$$\left.-\ldots-\binom{n}{n-1}F(\underbrace{y}_{1\text{-times}},\underbrace{y,y,\ldots,y}_{(n-1)\text{-times}})\right)\in Z(R).$$

This implies, $(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots$

$+\binom{n}{n-1})F(xy,xy,\ldots,xy)-(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$

$+\ldots+\binom{n}{n-1})F(y,y,\ldots,y)\in Z(R)$.

This gives,

$(2^n-2)F(xy,xy,\ldots,xy)-(2^n-2)F(y,y,\ldots,y)\in Z(R)$ for all $x,y\in L$.

Therefore,

$(2^n-2)\big(F(xy,xy,\ldots,xy)-F(y,y,\ldots,y)\big)\in Z(R)$ for all $x,y\in L$. (3.20)

Since R is not of characteristic (2^n-2), we find $\big(F(xy,xy,\ldots,xy)-F(y,y,\ldots,y)\big)\in Z(R)$ i.e. $f(xy)-f(y)\in Z(R)$ for all $x,y\in L$. Using (3.19), we find that $[x,y]\in Z(R)$ for all $x,y\in L$. Using Lemma 2.1, we obtain $L\subseteq Z(R)$.

Similarly, we can prove the result if $f(xy)+f(y)+[x,y]\in Z(R)$ for all $x,y\in L$.

Theorem 3.11. Let R be a semiprime ring of characteristic not $(2^n-2)$and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f([x,y])\mp f(x)\mp[x,y]\in Z(R)$ for all $x,y\in L$, then $L\subseteq Z(R)$.

Proof. Suppose,

$f([x,y])-f(x)-[x,y]\in Z(R)\ for\ all\ x,y\in L$. (3.21)

Replacing x by $x+z$ in (3.21), we obtain,

$f([x,y])+f([z,y])+\sum_{k=1}^{n-1}\binom{n}{k}h_k([x,y],[z,y])-f(x)-f(z)$

$-\sum_{k=1}^{n-1}\binom{n}{k}h_k(x,z)-[x,y]-[z,y]\in Z(R)$. (3.22)

Using (3.22), we have,

$\sum_{k=1}^{n-1}\binom{n}{k}h_k([x,y],[z,y])-\sum_{k=1}^{n-1}\binom{n}{k}h_k(x,z)\in Z(R)$. (3.23)

Substituting x for z in (3.24), we get,

$\sum_{k=1}^{n-1}\binom{n}{k}h_k([x,y],[x,y])-\sum_{k=1}^{n-1}\binom{n}{k}h_k(x,x)\in Z(R)$.

This can be written as,

$(\binom{n}{1}h_1([x,y],[x,y])$

$+\binom{n}{2}h_2([x,y],[x,+\binom{n}{3}h_3([x,y],[x,y])+\cdots$

$+\binom{n}{n-1}h_{n-1}([x,y],[x,y])$

-$\binom{n}{1}h_1(x,x)-\binom{n}{2}h_2(x,x)$

-$\binom{n}{3}h_3(x,x)-\ldots-\binom{n}{n-1}h_{n-1}(x,x))\in Z(R)$.

that is,

$$\left(\binom{n}{1}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-1)\text{-times}},\underbrace{[x,y])}_{1\text{-times}}\right.$$
$$+\binom{n}{2}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-2)\text{-times}},\underbrace{[x,y])}_{2\text{-times}}$$
$$+\binom{n}{3}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-3)\text{-times}},\underbrace{[x,y])}_{3\text{-times}}$$
$$+\ldots+\binom{n}{n-1}F(\underbrace{[x,y],}_{1\text{-times}}\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-1)\text{-times}})$$

$$-\binom{n}{1}F(\underbrace{x,x,\ldots,x}_{(n-1)\text{-times}},\underbrace{x}_{1\text{-times}})$$
$$-\binom{n}{2}F(\underbrace{x,x,\ldots,x}_{(n-2)\text{-times}},\underbrace{x}_{2\text{-times}})$$
$$-\binom{n}{3}F(\underbrace{x,x,\ldots,x}_{(n-3)\text{-times}},\underbrace{x}_{3\text{-times}})$$

$$-\cdots-\binom{n}{n-1}F(\underbrace{x}_{1\text{-times}},\underbrace{x,x,\ldots,x}_{(n-1)\text{-times}})\in Z(R).$$

This implies,

$((\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots$

$+\binom{n}{n-1})F([x,y],[x,y],\ldots,[x,y])-(\binom{n}{1}+\binom{n}{2}$

$+\binom{n}{3}+\ldots+\binom{n}{n-1})F(x,x,\ldots,x)\in Z(R)$.

This gives,

$(2^n-2)F([x,y],[x,y],\ldots,[x,y])-(2^n-2)F(x,x,\ldots,x)\in Z(R)$ for all $x,y\in L$.

Therefore,

$(2^n-2)\big(F([x,y],[x,y],\ldots,[x,y])-F(x,x,\ldots,x)\big)\in Z(R)$ for all $x,y\in L$.

Since R is not of characteristic (2^n-2), we find $\big(F([x,y],[x,y],\ldots,[x,y])-F(x,x,\ldots,x)\big)\in Z(R)$ for all $x,y\in L$. This implies that,

$f([x,y])-f(x)\in Z(R)\ for\ all\ x,y\in L.$ (3.24)

Using (3.22), (3.25) yields that $[x,y]\in Z(R)$ for all $x,y\in L$. This implies that $[L,L]\subseteq Z(R)$. Now using Lemma 2.1 we get the result.

Similarly, we can prove the result for the case $f([x,y])+f(x)+[x,y]\in Z(R)\ for\ all\ x,y\in L$.

Theorem 3.12. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f([x,y])\mp f(y)\mp[x,y]\in Z(R)$ for all $x,y\in L$, then $L\subseteq Z(R)$.

Proof. Suppose,

$f([x,y])-f(y)-[x,y]\in Z(R)$ for all $x,y\in$ L. (3.25)

Replacing y by $y+z$ in (3.26), we obtain,

$f([x,y])+f([x,z])+\sum_{k=1}^{n-1}\binom{n}{k}h_k([x,y],[x,z])-f(y)$

$-f(z)-\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)-[x,y]-[x,z]\in Z(R)$. (3.26)

Using (3.26) and (3.27), yields that

$\sum_{k=1}^{n-1}\binom{n}{k}h_k([x,y],[x,z])-\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\in Z(R)$. (3.27)

Substituting y for z in (3.28),

we get $\sum_{k=1}^{n-1}\binom{n}{k}h_k([x,y],[x,y])-\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,y)\in Z(R)$.

This can be written as,

$(\binom{n}{1}h_1([x,y],[x,y])+\binom{n}{2}h_2([x,y],[x,y])$

$+\binom{n}{3}h_3([x,y],[x,y])+\ldots+\binom{n}{n-1}h_{n-1}([x,y],[x,y])$

$-\binom{n}{1}h_1(y,y)-\binom{n}{2}h_2(y,y)$

$-\binom{n}{3}h_3(y,y)-\ldots-\binom{n}{n-1}h_{n-1}(y,y))\in Z(R)$

that is,

$$\Bigg(\binom{n}{1}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-1)\text{-times}},\underbrace{[x,y]}_{1\text{-times}})$$

$$+\binom{n}{2}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-2)\text{-times}},\underbrace{[x,y]}_{2\text{-times}})$$

$$+\binom{n}{3}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-3)\text{-times}},\underbrace{[x,y]}_{3\text{-times}})$$

$$+\cdots+\binom{n}{n-1}F(\underbrace{[x,y]}_{1\text{-times}},\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-1)\text{-times}})$$

$$-\binom{n}{1}F(\underbrace{y,y,\ldots,y}_{(n-1)\text{-times}},\underbrace{y}_{1\text{-times}})$$

$$-\binom{n}{2}F(\underbrace{y,y,\ldots,y}_{(n-2)\text{-times}},\underbrace{y}_{2\text{-times}})$$

$$-\binom{n}{3}F(\underbrace{y,y,\ldots,y}_{(n-3)\text{-times}},\underbrace{y}_{3\text{-times}})$$

$$-\cdots-\binom{n}{n-1}F(\underbrace{y}_{1\text{-times}},\underbrace{y,y,\ldots,y}_{(n-1)\text{-times}})\Bigg)\in Z(R).$$

This implies,

$((\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots$

$+\binom{n}{n-1})F([x,y],[x,y],\ldots,[x,y])-(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$

$+\ldots+\binom{n}{n-1})F(y,y,\ldots,y)\in Z(R)$.

This gives,

$(2^n-2)F([x,y],[x,y],\ldots,[x,y])-(2^n-2)F(y,y,\ldots,y)\in Z(R)$ for all $x,y\in L$.

Therefore,

$(2^n-2)\big(F([x,y],[x,y],\ldots,[x,y])-F(y,y,\ldots,y)\big)\in Z(R)$ for all $x,y\in L$.

Since R is not of characteristic (2^n-2), we find $(F([x,y],[x,y],\ldots,[x,y])-F(y,y,\ldots,y))\in Z(R)$ for all $x,y\in L$. This implies that $f([x,y])-f(y)\in Z(R)$ for all $x,y\in L$. Using (3.26), $[x,y]\in Z(R)\ for\ all\ x,y\in L$.

This implies that $[L,L] \subseteq Z(R)$. Now using Lemma 2.1, we have $L \subseteq Z(R)$.

Similarly, we can prove the result for the case $f([x,y]) + f(y) + [x,y] \in Z(R)$ for all $x,y \in L$.

Using the similar techniques as we have used in the proof of Theorem 3.11 and Theorem 3.12, we can prove the following:

Theorem 3.13. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f([x,y]) \mp f(x) \mp [y,x] \in Z(R)\, for\, all\, x,y \in L$, then $L \subseteq Z(R)$

Theorem 3.14. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F.

If $f([x,y]) \mp f(y) \mp [y,x] \in Z(R)$ for all $x,y \in L$,

then $L \subseteq Z(R)$.

Theorem 3.15. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-times}}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f([x,y]) \mp f(xy) \mp [x,y] \in Z(R)$ for all $x,y \in L$, then $L \subseteq Z(R)$.

Proof. Suppose,

$f([x,y]) - f(xy) - [x,y] \in Z(R)$ for all $x,y \in L$. (3.28)

Replacing y by $y+z$ in (3.28), we get,

$f([x,y]) + f([x,z])$

$+\sum_{k=1}^{n-1}\binom{n}{k} h_k([x,y],[x,z]) - f(xy)$

$- f(xz) - \sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xz) - [x,y] - [x,z] \in Z(R)$. (3.29)

Using (3.28) and (3.29), we obtain,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k([x,y],[x,z]) - \sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xz) \in Z(R)\, for\, all\, x,y,z \in L$. (3.30)

Substituting y for z in (3.30), we get,

$\sum_{k=1}^{n-1}\binom{n}{k} h_k([x,y],[x,y]))$

$-\sum_{k=1}^{n-1}\binom{n}{k} h_k(xy,xy) \in Z(R)f$

for all $x,y \in L$.

This can be written as,

$((\binom{n}{1}h_1([x,y],[x,y]) + \binom{n}{2}h_2([x,y],[x,y])$

$+\binom{n}{3}h_3([x,y],[x,y]) + \cdots + \binom{n}{n-1}h_{n-1}([x,y],[x,y])$

$-\binom{n}{1}h_1(xy,xy) - \binom{n}{2}h_2(xy,xy)$

$-\binom{n}{n-1}h_3(xy,xy) - \ldots - \binom{n}{n-1}h_{n-1}(xy,xy)) \in Z(R)$

that is,

$$\left(\binom{n}{1}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-1)\text{-times}},\underbrace{[x,y])}_{1\text{-times}}\right.$$

$$+\binom{n}{2}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-2)\text{-times}},\underbrace{[x,y])}_{2\text{-times}}$$

$$+\binom{n}{3}F(\underbrace{[x,y],[x,y],\ldots,[x,y]}_{(n-3)\text{-times}},\underbrace{[x,y])}_{3\text{-times}}$$

$$+\cdots+\binom{n}{n-1}F(\underbrace{[x,y]}_{1\text{-times}},\underbrace{[x,y],[x,y],\ldots,[x,y])}_{(n-1)\text{-times}}$$

$$-\binom{n}{1}F(\underbrace{xy,xy,\ldots,xy}_{(n-1)\text{-times}},\underbrace{xy}_{1\text{-times}})$$

$$-\binom{n}{2}F(\underbrace{xy,xy,\ldots,xy}_{(n-2)\text{-times}},\underbrace{xy}_{2\text{-times}})$$

$$-\binom{n}{3}F(\underbrace{xy,xy,\ldots,xy}_{(n-3)\text{-times}},\underbrace{xy}_{3\text{-times}})$$

$$\left.-\cdots-\binom{n}{n-1}F(\underbrace{xy}_{1\text{-times}},\underbrace{xy,xy,\ldots,xy}_{(n-1)\text{-times}})\right)\in Z(R).$$

This implies,

$\left(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n-1}\right)F([x,y],[x,y],\ldots,[x,y])$

$-(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\ldots+\binom{n}{n-1})F(xy,xy,\ldots,xy) \in Z(R)$.

Thus,

$(2^n-2)F([x,y],[x,y],\ldots,[x,y]) - (2^n-2)F(xy,xy,\ldots,xy) \in Z(R)$ for all $x,y \in L$.

Therefore,

$(2^n-2)\big(F([x,y],[x,y],\ldots,[x,y]) - F(xy,xy,\ldots,xy)\big) \in Z(R)$ for all $x,y \in L$.

Since R is not of characteristic (2^n-2), we find $(F([x,y],[x,y],\ldots,[x,y]) - F(xy,xy,\ldots,xy)) \in Z(R)\, for\, all\, x,y \in L$. This implies that $f([x,y]) - f(xy) \in Z(R)\, for\, all\, x,y \in L$. Using (3.28), we have $[x,y] \in Z(R)\, for\, all\, x,y \in L$. This implies that $[L,L] \subseteq Z(R)$. Now using Lemma 2.1, we have $L \subseteq Z(R)$.

Similarly, we can prove the result if $f([x,y]) + f(xy) + [x,y] \in Z(R)\, for\, all\, x,y \in L$.

Theorem 3.16. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\ldots\times R}_{n\text{-}times}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f([x,y])\mp f(xy)\mp [y,x]\in Z(R)$ for all $x,y\in L$, then $L\subseteq Z(R)$.

Theorem 3.17. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\ldots\times R}_{n\text{-}times}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f(x)f(y)\mp [x,y]\in Z(R)$ for all $x,y\in L$, then $L\subseteq Z(R)$.

Proof. Let

$f(x)f(y)-[x,y]\in Z(R)$ for all $x,y\in L$. (3.31)

Replacing $y+z$ by y in (3.31), we get,

$f(x)f(y)+f(x)f(z)+f(x)\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)-[x,y]-[x,z]\in Z(R)$. (3.32)

Using (3.31), we obtain,

$$f(x)\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\in Z(R).$$

This can be written as,

$f(x)\left(\binom{n}{1}h_1(y,z)+\binom{n}{2}h_2(y,z)+\binom{n}{3}h_3(y,z)+\ldots+\binom{n}{n-1}h_{n-1}(y,z)\right)\in Z(R)$. (3.33)

Substituting z by y in (3.33), we get,

$$f(x)\left(\binom{n}{1}F(\underbrace{y,y,\ldots,y}_{(n-1)\text{-}times},\underbrace{y}_{1\text{-}times})+\binom{n}{2}F(\underbrace{y,y,\ldots,y}_{(n-2)\text{-}times},\underbrace{y}_{2\text{-}times})+\binom{n}{3}F(\underbrace{y,y,\ldots,y}_{(n-3)\text{-}times},\underbrace{y}_{3\text{-}times})+\ldots+\binom{n}{n-1}F(\underbrace{y}_{1\text{-}times},\underbrace{y,y,\ldots,y}_{(n-1)\text{-}times})\right)\in Z(R)$$

for all $x,y\in L$.

that is,

$f(x)\left(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\ldots+\binom{n}{n-1}\right)F(y,y,\ldots,y)\in Z(R)$ for all $x,y\in L$.

This implies,

$(2^n-2)f(x)F(y,y,\ldots,y)\in Z(R)$ for all $x,y\in L$.

Since R is not of characteristic (2^n-2), we have $f(x)F(y,y,\ldots,y)\in Z(R)\ for\ all\ x,y\in L$. This implies that $f(x)f(y)\in Z(R)$.

Using (3.31), we obtain $[x,y]\in Z(R)\ for\ all\ x,y\in L$. i.e. $[L,L]\subseteq Z(R)$.

Application of Lemma 2.1,we get the result.

Similarly, we can prove the result if $f(x)f(y)+[x,y]\in Z(R)\ for\ all\ x,y\in L$.

Theorem 3.18. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\ldots\times R}_{n\text{-}times}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f(x)f(y)\mp [y,x]\in Z(R)\ for\ all\ x,y\in L$, then $L\subseteq Z(R)$.

Proof. The proof runs on the same parallel lines as that of Theorem 3.17.

Theorem 3.19. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\ldots\times R}_{n\text{-}times}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f(x)f(y)\mp xy\in Z(R)$ for all $x,y\in L$, then $L\subseteq Z(R)$.

Proof. Let

$f(x)f(y)-xy\in Z(R)\ for\ all\ x,y\in L$. (3.34)

Replacing $y+z$ by y in (3.34), we have,

$f(x)f(y)+f(x)f(z)+f(x)\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)-xy-xz\in Z(R)\ for\ all\ x,y,z\in L$. (3.35)

Applying (3.34), we obtain,

$f(x)\sum_{k=1}^{n-1}\binom{n}{k}h_k(y,z)\in Z(R)\ for\ all\ x,y,z\in L$.

This can be written as,

$f(x)\left(\binom{n}{1}h_1(y,z)+\binom{n}{2}h_2(y,z)+\binom{n}{3}h_3(y,z)+\ldots+\binom{n}{n-1}h_{n-1}(y,z)\right)\in Z(R)$. (3.36)

Substituting z by y in (3.36), we get,

$$f(x)\left(\binom{n}{1}F(\underbrace{y,y,\ldots,y}_{(n-1)\text{-}times},\underbrace{y}_{1\text{-}times})+\binom{n}{2}F(\underbrace{y,y,\ldots,y}_{(n-2)\text{-}times},\underbrace{y}_{2\text{-}times})+\binom{n}{3}F(\underbrace{y,y,\ldots,y}_{(n-3)\text{-}times},\underbrace{y}_{3\text{-}times})+\ldots+\binom{n}{n-1}F(\underbrace{y}_{1\text{-}times},\underbrace{y,y,\ldots,y}_{(n-1)\text{-}times})\right)\in Z(R)$$

for all $x,y\in L$

that is,

$f(x)\left(\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\ldots+\binom{n}{n-1}\right)F(y,y,\ldots,y)\in Z(R)$ for all $x,y\in L$.

This implies,

$(2^n-2)f(x)F(y,y,\ldots,y)\in Z(R)$ for all $x,y\in L$.

Since R is not of characteristic (2^n-2). Then $f(x)F(y,y,\ldots,y)\in Z(R)$ for all $x,y\in L$. This implies that $f(x)f(y)\in Z(R)$.

Using (3.34), implies that $xy\in Z(R)$ *for all* $x,y\in L$. Application of Lemma 2.2, we get the result.

Similarly, we can prove the result if $f(x)f(y)+xy\in Z(R)$ *for all* $x,y\in L$.

Theorem 3.20. Let R be a semiprime ring of characteristic not (2^n-2) and L be a nonzero Lie ideal of R. Let $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$ be a symmetric n-additive mapping and f be the trace of F. If $f(x)f(y)\mp yx\in Z(R)$ *for all* $x,y\in L$, then $L\subseteq Z(R)$.

The following examples illustrates that R to be semiprime and characteristic not (2^n-2) for n>1 is essential in the hypothesis of the above theorem.

Example 3.21. Let $R=\left\{\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}\middle|\, a,b,c\in Z, ring\ of\ integers\right\}$ and $L=\left\{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\middle|\, b\in Z\right\}$. Then $Z(R)=\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\middle|\, a\in Z\right\}$. Define map $F:\underbrace{R\times R\times\cdots\times R}_{n\text{-}times}\to R$ by

$F=\left(\begin{pmatrix}a_1 & b_1\\ 0 & c_1\end{pmatrix},\begin{pmatrix}a_2 & b_2\\ 0 & c_2\end{pmatrix},\ldots,\begin{pmatrix}a_n & b_n\\ 0 & c_n\end{pmatrix}\right)=\begin{pmatrix}a_1a_2a_3\ldots a_n & 0\\ 0 & 0\end{pmatrix}$.

It can be verified that F is n-additive with trace f defined by $f:R\to R$ such that

$$f\left(\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}\right)=F\left(\begin{pmatrix}a & b\\ 0 & c\end{pmatrix},\begin{pmatrix}a & b\\ 0 & c\end{pmatrix},\ldots,\begin{pmatrix}a & b\\ 0 & c\end{pmatrix}\right)$$

satisfying following hypothesis of the above theorems. However, L $\nsubseteq Z(R)$.

REFERENCES

[1] Ashraf M, Ali A and Ali S. Some commutativity theorems for rings with generalized derivations. Southeast Asian Bull Math 2007; 31: 415-421.

[2] Bell HE and Martindale WS. Centralizing mappings of semiprime rings. Canad Math Bull 1987; 30: 92-101.

[3] Daif MN and Bell HE. Remarks on derivations on semiprime rings. Int J Math and Math Sci 1992; 15 (1): 205-206.

[4] Herstien IN. On Lie structure of an associative ring. J Algebra 1970; 14 (4): 561-571.

[5] Park KH. On prime and semiprime rings with symmetric n-derivations. J Chungcheong Math Soc 2009; 22: 451-458.

[6] Park KH and Jung YS. On permuting 3-derivations and commutativity in prime near rings. Commun Korean Math Soc 2010; 25: 1-9, DOI 10.4134/CKMS. 2010. 25. 1001.

[7] Maksa G. A remark on symmetric biadditive functions having non-negative diagonalization. Glas Math Ser III 1980; 15: 279-280.

[8] Lee PH and Wang RJ. Commuting traces of multiadditive mappings. J Algebra 1997; 193: 709-723.

[9] Bresar M. Commuting traces of biadditive mappings commutativity-preserving mappings and Lie mappings. Trans Amer Math Soc 1993; 335 (2).

Received on 03-01-2017 Accepted on 26-05-2017 Published on 30-07-2017

DOI: http://dx.doi.org/10.15377/2409-5761.2017.04.01.1