Matrix Transforms by Factorable Matrices

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Abstract: In the present paper an overview of existing results on matrix transforms of summability and absolute summability domains of matrix methods by factorable matrices is presented. Under the notion “multiplicative matrix” we consider a lower triangular matrix $M = (m_{nk})$, where $m_{nk} = r_n v_k$ with $r_n, v_k \in C.$

Keywords: Matrix transforms, factorable matrices, conservative and regular matrices, Riesz matrix.

1. INTRODUCTION

In this paper matrix transforms of sequence spaces by factorable matrices are investigated. Throughout in the present paper by *M* we denote a factorable matrix; i.e., $M = (m_{nk})$ is a lower triangular matrix, where,

$$m_{nk} = r_n v_k, \quad k \le n; \quad r_n, v_k \in C.$$

The set of all factorable matrices we denote by *F*. Also throughout this paper, we assume that indices and summation indices run from 0 to ∞ unless otherwise specified. Let ω be the set of all sequences with real or complex entries, $m \subset \omega$ the set of all bounded sequences, $c \subset m$ the set of all convergent sequences, $c_0 \subset c$ the set of all null sequences,

$$cs := \left\{ x = (x_k) : \exists \lim_n \sum_{k=0}^{n} x_k \right\},$$

$$l := \left\{ x = (x_k) : \sum_{k=0}^{n} |x_k| < \infty \right\},$$

$$bv := \left\{ x = (x_k) : (\Delta x_k) \in l \right\}, \quad \Delta x_k := x_k - x_{k+1}.$$

Moreover, let $A = (a_{nk})$ be a matrix with real or complex entries and;

$$A_n x := \sum_k a_{nk} x_x, \quad Ax := \left(A_n x \right)$$

for every $x = (x_k) \in \omega$. Let X, Y be some subsets of ω and;

$$X_A := \left\{ x = (x_k) \in \omega : Ax \in X \right\},$$

$$(X, Y) := \left\{ A = (a_{nk}) : Ax \in Y \text{ for every } x \in X \right\}.$$

A matrix A is called reversible if the infinite system of equations $z_n = A_n x$ has a unique solution for each sequence $(z_n) \in c,$ and normal if A is lower triangular with $a_{nn} \ne 0.$ Necessary and sufficient conditions for $Y \subseteq X_A$ if $X, Y = c, c_0, cs, l, bv$ have been widely investigated; referring only to monographs [1], [12] - [15] and [37], a good overview has been given also in [35]. Also the inclusion $X_A \subseteq Y_B$ for $X, Y = c, c_0, cs, l, bv$ (B is a matrix with real or complex entries) has been well investigated for reversible or normal A; see also [1], [12] - [15] and [37]. Necessary and sufficient conditions for a matrix $D \in (X_A, Y_B)$ if $X, Y = c, c_0, cs, l, bv$ in case of reversible or normal A have been presented, for example, in papers [2], [5]-[7], [10], [11], [36], and in textbook [1]. Often these necessary and sufficient conditions are difficult to check. Due to the simple structure, better controlled conditions can be obtained for a factorable matrix M. In this paper, we give an overview of the known results on matrix transformations by factorable matrices; we do not consider the topological properties of factorable matrices. Note that the topological properties of factorable matrices can be found, for example, from papers [16]-[26], [28], [29], [31-34] and [38].

The paper is organized as follows. In Section 2, some examples of factorable matrices have been introduced. In Section 3, the summability domain c_M for $M \in F$ has been described. In Section 4, necessary and sufficient conditions for $l_A \subset cs_M$ and $l_A \subset l_M$ for a normal matrix A have been presented. In Section 5, necessary and sufficient conditions for $M \in (c_A, c_B)$, if A is the Cesàro method C^α of order α; $\alpha \in C; \alpha \ne -1, -2, \ldots$, have been described.

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2. SOME WELL-KNOWN SUMMABILITY METHODS DEFINED BY FACTORABLE MATRICES

First we introduce the class of normal factorable matrices.

Theorem 2.1 ([9], p. 2-3)**.** *A normal matrix* $A = (a_{nk})$ *is factorable if and only if its inverse is bidiagonal.*

Proof. Necessity. Let *A* be a normal factorable matrix; i.e., $a_{nk} = r_n v_k$. Then it is easy to find that $A^{-1} = (a_{nk}^{-1})$ is a normal matrix with;

$$a_{nk}^{-1} = \begin{cases} 1/r_n v_n & \text{if } k = n, \\ -1/r_{n-1} v_n & \text{if } k = n-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Hence A^{-1} is bidiagonal.

Sufficiency. Let *A* be a normal matrix with a bidiagonal inverse $A^{-1} = (a_{nk}^{-1})$; i.e.,

$$a_{nn}^{-1} = \alpha_n, \ \ a_{n,n-1}^{-1} = \beta_{n-1} (n \geq 1),$$

$$a_{nk}^{-1} = 0 \ \text{ for } \ 0 \leq k < n-1.$$

Then $a_{nn} = 1/\alpha_n$, and;

$$a_{n,n-1} a_{n-1,n-1}^{-1} + a_{nn} a_{n,n-1}^{-1} = 0.$$

This implies;

$$a_{n,n-1} = -\frac{\beta_{n-1}}{\alpha_{n-1}\alpha_n},$$

for $n \geq 1$. Now with the help of mathematical induction it is possible to show that;

$$a_{nk} = (-1)^{n-k} \frac{\prod_{j=k+1}^{n} \beta_{j-1}}{\prod_{j=k}^{n} \alpha_j}$$

$$= (-1)^n \frac{\prod_{j=0}^{n-1} \beta_j}{\prod_{j=0}^{n} \alpha_j} (-1)^k \frac{\prod_{j=0}^{k-1} \alpha_j}{\prod_{j=0}^{k-1} \beta_j}$$

for $k \leq n-1$ and $n \geq 1$. Thus *A* is factorable.

Further we introduce some well-known matrix methods of summability, which are defined by factorable matrices.

2.1. A weighted Mean Method of Riesz $(\mathbf{R}, \mathbf{p_k})$

A summability method (R, p_k) is defined by a lower triangular infinite matrix $A = (a_{nk})$ with $a_{nk} = p_k / P_n$, where;

$$p_0 > 0, \ \ p_k \geq 0 \ \text{ and } \ P_n = \sum_{k=0}^{n} p_k.$$

It is easy to see that (R, p_k) is a special case of a factorable matrix obtained by setting $v_k = p_k$ and $r_n = 1/P_n$.

2.2. Method of Cesàro $\mathbf{C}^1$ of Order One

The method C^1 is a special case of (R, p_k), where $p_k = 1$; i.e., $C^1 = (a_{nk})$ is a lower triangular infinite matrix with,

$$a_{nk} = \frac{1}{n+1}; k \leq n.$$

2.3. *p*-Cesàro Method $(\mathbf{C}, \mathbf{p})$ of Order One

(C, p) is defined by a lower triangular infinite matrix $A = (a_{nk})$ with (see [16], p.127),

$$a_{nk} = \frac{1}{(n+1)^p}; k \leq n, \ \ p > 0.$$

Indeed, setting $v_k \equiv 1$ and $r_n = 1/(n+1)^p$, we see that (C, p) is factorable.

2.4. Generalized Cesàro Method $(\mathbf{C}, \mathbf{1}, \mathbf{i})$ of Order One

$(C, 1, k)$ is defined by a lower triangular infinite matrix $A = (a_{nk})$ with (see [16], p.127-128),

$$a_{nk} = \frac{1}{n+i}; k \leq n, \quad i > 0.$$

Taking $v_k \equiv 1$ and $r_n = 1/n + i$, we see that $(C, 1, i)$ is factorable.

2.5. H-J Generalized Hausdorff Matrices

Let (λ_n) be a strictly increasing sequence of real numbers satisfying the properties,

$$0 \le \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots,$$

$$\lim_n \lambda_n = \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

Such a sequence (λ_n) we shall call admissible. Let (μ_n) be a sequence of real numbers. The generalized Hausdorff matrix, shortly H-J matrix, is defined by $H = (h_{nk})$,

$$h_{nk} = \lambda_{k+1} \ldots \lambda_n [\mu_k, \ldots, \mu_n], \quad k \le n,$$

where [ ] is the divided difference defined by;

$$[\mu_k, \mu_{k+1}] := \frac{\mu_k - \mu_{k+1}}{\lambda_{k+1} - \lambda_k}$$

and,

$$[\mu_k, \ldots, \mu_n] := \frac{[\mu_k, \ldots, \mu_{n-1}] - [\mu_{k+1}, \ldots, \mu_n]}{\lambda_n - \lambda_k},$$

with the understanding that the product $\lambda_{k+1} \ldots \lambda_n = 1$ if $k = n$ (see [8-9]).

It has been shown in [9] that under certain conditions, a conservative (or regular) H-J matri**x** is factorable.

Let us remember that a matrix *A* is said to be conservative if $Ax \in c$ for every $x = (x_n) \in c$, and regular if $\lim_n A_n x = \lim_n x_n$ for every $x \in c$. It is known (see [8-9] that an H-J matrix is conservative if and only if there exists a function of bounded variation χ over $[0;1]$, such that;

$$\int_0^1 |d\chi(x)| < \infty, \tag{2.2}$$

where the integral is a Riemann-Stieltjes one. Moreover,

$$\mu_n = \int_0^1 x^{\lambda_n} d\chi(x).$$

Theorem 2.2 ([9], p. 3-6). *Let H be a conservative H-J matrix with* $\lambda_0 = 0$. *Then H is factorable if and only if;*

$$\mu_n = \frac{a}{\lambda_n + a}, \text{where } a = \frac{\mu_1 \lambda_1}{1 - \mu_1},$$

or $\mu_0 = 1,\ \mu_n = 0$ *for all* $n > 0$.

Theorem 2.3 ([9], p. 6-7). *Let H be a normal conservative H-J matrix with* $\lambda_0 > 0$. *Then H is factorable if and only if;*

$$\mu_n = \frac{\mu_0 b}{\lambda_n - \lambda_0 + b}, \text{ where } a = \frac{\mu_1(\lambda_1 - \lambda_0)}{\mu_0 - \mu_1} > \lambda_0.$$

2.6. E-J Matrices

E-J matrix is defined by $E^{(\alpha)} = (e_{nk}^{(\alpha)})$, where (see [8-9];

$$e_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \quad \alpha > 0;$$

$\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\Delta^{n+1}\mu_k = \Delta(\Delta^n \mu_k)$.

Necessary and sufficient condition for an E-J matrix to be conservative is the existence of a function of bounded variation χ over $[0;1]$, such that (2.2) is satisfied. For the E-J matrices the diagonal entries take the form ([9], p.7),

$$e_{nn}^{(\alpha)} = \int_0^1 x^{n+\alpha} d\chi(x).$$

It is easy to see that the E-J matrix is the special case of the H-J matrix with $\lambda_n = n + \alpha$.

Theorem 2.4 ([9], p. 6-7). *Let* $E^{(\alpha)} = (e_{nk}^{(\alpha)})$ *be a normal conservative E-J matrix. Then* $E^{(\alpha)}$ *is factorable if and only if;*

$$e_{nn}^{(\alpha)} = \frac{\mu_0 c}{n + c}, \text{ where } c = \frac{\mu_1}{\mu_0 - \mu_1}.$$

3. SUMMABILITY DOMAINS OF FACTORABLE MATRICES

In this section we consider factorable matrices with nonnegative entries; i.e., we consider the subset $F^+ \subset F$ defined as follows:

$$F^+ := \left\{ M = (r_n v_k) \in F : v_0, r_n > 0, v_k \ge 0, k = 1, 2, \ldots \right\}.$$

We describe the summability domains of $M \in F^+$ *via* (R, p_k). For this purpose for each factorable matrix $M = (r_n v_k) \in F$ we use its associated Riesz matrix (R, v_k).

First we present some auxiliary notions and results.

Lemma 3.1 ([13], *Theorem* 2.3.7 *or* [35], *Propositions* 11 and 23). *A matrix* $A=(a_{nk})$ *is conservative if and only if;*

there exist finite limits $\lim_n a_{nk} := s_k,$ (3.1)

there exist finite limits $\lim_n \sum_k a_{nk} := t,$ (3.2)

$$\sum_k |a_{nk}| = O(1).$$

A matrix A is regular if and only if conditions (3.1) – (3.3) are satisfied and $s_k \equiv 0$, $t=0.$ *A matrix A is regular for* c_0 *if and only if conditions (3.1) and (3.3) are satisfied and* $s_k \equiv 0.$

A matrix $A=(a_{nk})$ is said to be coercive if $m \subset c_A.$

Lemma 3.2 ([13], Theorem 2.4.1 or [35], Proposition 10). *A matrix* $A=(a_{nk})$ *is coercive if and only if conditions (3.1) and (3.3) are fulfilled and*

$$\lim_n \sum_k |a_{nk} - s_k| = 0.$$

Lemma 3.3 ([13], p. 51). *A coercive matrix cannot be regular.*

From Lemma 3.1 it is easy to conclude that $M \in F^+$ is conservative if and only if there exist the finite limits;

$$\lim_n r_n := r, \quad \lim_n r_n V_n := q; \; V_n := \sum_{k=0}^{n} v_k,$$

and $M \in F^+$ is regular if $r=0$ and $q=1.$

A conservative matrix $A=(a_{nk})$ is said to be coregular if;

$$\rho(A) := \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} \neq 0,$$

and conull if $\rho(A)=0.$

Lemma 3.4 *A conservative matrix* $M=(r_n v_k) \in F^+$ *is coregular if and only if* $r=0$ *and* $q \neq 0.$

Proof. Necessity. Let $r=0$ and $q \neq 0.$ Then obviously $\rho(M) \neq 0.$

Sufficiency. Let $\rho(M) \neq 0.$ Then;

$$\lim_n r_n V_n \neq \sum_k \left(\lim_n r_n\right) v_k$$

or

$$\lim_n r_n V_n \neq r \lim_n V_n. \tag{3.3}$$

If $r \neq 0,$ then there exists the finite limit $\lim_n V_n,$ since, due to conservativity of *M*, $\lim_n r_n V_n$ exists. Hence, $\lim_n r_n V_n = r \lim_n V_n,$ which is in contradiction with (3.3). Thus $r=0$ and $q=\lim_n r_n V_n \neq 0.$

We note that Lemma 3.4 was given in [27] without proof.

Theorem 3.5 ([38], p. 380). *A conservative matrix* $M=(r_n v_k) \in F^+$ *is either coregular or coercive.*

Proof. If $q=\lim_n r_n V_n = 0,$ then,

$$r_n = r_n V_n \cdot \frac{1}{V_n} \leq r_n V_n \cdot \frac{1}{V_0} \to 0.$$

This implies that M is coercive.

If $q=\lim_n r_n V_n \neq 0$ and $r=\lim_n r_n = 0,$ then *M* is coregular by Lemma 3.4. If $q \neq 0$ and $r \neq 0,$ then $\lim_n V_n := V < \infty.$ Hence $\lim_n (r_n - r) V_n = 0 \cdot V = 0.$ Therefore M is coercive by Lemma 3.2.

Theorem 3.6 ([38], p. 380-381). *Let* $M=(r_n v_k) \in F^+$ *be conservative. Then the following assertions hold:*

(i) $c_{(R,v_k)} \subset c_M$ *and;*

$$\lim_n M_n x = q \lim_n (R, v_k)_n x \tag{3.4}$$

for every $x \in c_{(R,v_k)}.$

(ii) *If* $v_k \neq 0$ *for infinite numbers of indices k, then* $c_M \subset c_{(R,v_k)}$ if and only if $q \neq 0.$

Proof. (i) As *M* is conservative and,

$$M_n x = r_n \sum_{k=0}^{n} v_k x_k = r_n V_n \cdot \frac{1}{V_n} \sum_{k=0}^{n} v_k x_k = r_n V_n (R, v_k)_n x \tag{3.5}$$

for every $x \in c_{(R,v_k)}$, then $x \in c_M$. Moreover, relation (3.4) holds for every $x \in c_{(R,v_k)}$ by (3.5).

(ii) If $q = \lim_n r_n V_n \neq 0$, then the inclusion $c_M \subset c_{(R,v_k)}$ follows from (3.5). Assume that $q = 0$. We show that then $c_M \not\subset c_{(R,v_k)}$. Let (n_i) $\left(i \in N\right)$ be the set of indices, such that $v_{n_i} \neq 0$ and $v_n = 0$ for $n \notin (n_i)$. Let;

$$k_i := \min\left\{ n_i \le \tau < n_{i+1} : r_\tau = \max\left\{r_k : n_i \le k < n_{i+1}\right\}\right\}.$$

Now we define inductively a sequence $\bar{x} = (x_n) \in c_M$ by setting $x_0 := 1$ and;

$$x_n := \begin{cases} \frac{1}{v_n}\left(\sqrt{\frac{V_{k_i}}{r_{k_i}}} - \sum_{k=0}^{n-1} v_k x_k\right) & \text{if } n = n_i \text{ for some } i \in N, \\ 0 & \text{if } n \notin \left\{n_i : i \in N\right\}. \end{cases}$$

Then we obtain,

$$M_{n_i}\bar{x} = r_{n_i}\left(\sqrt{\frac{V_{k_i}}{r_{k_i}}} - \sum_{k=0}^{n_i-1} v_k x_k + \sum_{k=0}^{n_i-1} v_k x_k\right) = \frac{r_{n_i}}{r_{k_i}}\sqrt{V_{k_i} r_{k_i}} \tag{3.6}$$

for every $i \in N$. As $\lim_n r_n V_n = 0$ and $V_n \ge v_0 > 0$, then $r = 0$. This implies $r_{n_i} / r_{k_i} = O(1)$, since r_n>0. Hence $\lim_i M_{n_i}\bar{x} = 0$ by (3.6). For $n_i < n < n_{i+1}$ we have;

$$M_n\bar{x} = r_{n_i}\sum_{k=0}^{n_i} v_k x_k = \frac{r_n}{r_{n_i}} M_{n_i}\bar{x} = \frac{r_n}{r_{n_i}}\frac{r_{n_i}}{r_{k_i}}\sqrt{V_{k_i} r_{k_i}} = \frac{r_n}{r_{k_i}}\sqrt{V_{k_i} r_{k_i}}.$$

Consequently $\lim_i M_n\bar{x} = 0$. On the other hand,

$$(R,v_k)_{k_i}\bar{x} = \frac{1}{V_{k_i} r_{k_i}} \cdot \frac{r_{k_i}}{r_{k_i}}\sqrt{V_{k_i} r_{k_i}} = \frac{1}{\sqrt{V_{k_i} r_{k_i}}},$$

by (3.5). Therefore $\lim_n (R,v_k)_{k_i}\bar{x} = \infty$; i.e., $\bar{x} \notin c_{(R,v_k)}$. So $c_M \not\subset c_{(R,v_k)}$.

As $r = 0$ and $q = 1$ for a regular method $M = (r_n v_k) \in F^+$, then from Theorem 3.6 we immediately obtain the following result.

Corollary 3.7. *Let* $M = (r_n v_k) \in F^+$ *be regular and* $v_k \neq 0$ *for infinite numbers of indices k. Then* $c_M = c_{(R,v_k)}$.

Theorem 3.8 ([38], p. 381). *Let* $M = (r_n v_k) \in F^+$ *be conservative. Then the following statements hold:*

(i) *If M is coregular, then* (R,v_k) *is regular.*

(ii) *If* (R,v_k) *is regular, then M is regular for* c_0.

Proof. (i) As $r = 0$, $q \neq 0$ and,

$$V_n = r_n V_n \cdot \frac{1}{r_n},$$

then $\lim_n V_n = \infty$; i.e., (R,v_k) is regular.

(ii) As $\lim_n V_n$ and *M* is conservative, then we obtain $r = \lim_n r_n = \lim_n \frac{1}{V_n} \cdot \lim_n r_n V_n = 0$; i.e., *M* is regular for c_0.

Theorem 3.9 (cf. [38], Proposition 2.5). *Let* $M = (r_n v_k) \in F^+$ *be conservative. Then the following statements hold:*

(i) *If* (R,v_k) *is coercive, then M is regular coercive.*

(ii) *If* $r \neq 0$, *then* (R,v_k) *is coercive.*

(iii) *If* $r = 0$, *then* (R,v_k) *can be both either coercive or regular.*

Proof. (i) As $V = \lim_n V_n = \infty$ for regular (R,v_k) and coercive (R,v_k) cannot be regular by Lemma 3.3, then $V < \infty$ for coercive (R,v_k). This implies that;

$$\lim_n (r_n - r)V_n = 0 \cdot V = 0. \tag{3.7}$$

Hence, *M* is coercive by Lemma 3.2.

(ii) As *M* is conservative, then there exists the limit $q = \lim_n r_n V_n < \infty$ by Lemma 3.1. Therefore, due to $r \neq 0$ and $v_0 > 0, v_k \ge 0$, we obtain that there exists the limit $V = \lim_n V_n$ with $0 \neq V < \infty$. This implies the existence of the finite limit $\lim_n \frac{1}{V_n} = \frac{1}{V}$. Hence (3.7) is fulfilled for $r_n = 1/V_n$. Consequently (R,v_k) is coercive by Lemma 3.2.

(iii) Let $r_n := 1/(n+1)^2$. If $V < \infty$, then both *M* and (R, v_k) are coercive. If, for example, $v_k \equiv 1$, then *M* is coercive and (R, v_k) is regular.

Remark 3.10. From the proof of Theorem 3.9, we can conclude that the assumption of coercivity of *M* in Proposition 2.5 of [38] is redundant.

Now we describe conservative matrices which are stronger than a given factorable matrix. We remember that a matrix *B* is said to be stronger than matrix *A* if $c_A \subset c_B$.

Theorem 3.11 ([38], Theorem 2.7). *Let* $M = (r_n v_k) \in F^+$ *be coregular with* $v_k > 0$. *Then a conservative matrix B is stronger than M if and only if;*

(i) $\left(\dfrac{b_{nk}}{v_k}\right) \in c_0$ *for every n,*

(ii) $\displaystyle\sum_k V_k \left| \frac{b_{nk}}{v_k} - \frac{b_{n,k+1}}{v_{k+1}} \right| = O(1).$

Proof. (i) Using Theorem 3.8 (i) we obtain that (R, v_k) is regular. In addition, $c_M = c_{(R,v_k)}$ by Theorem 3.6. Now the assertion of the theorem follows from Theorem 3.2.8 of [13].

From Theorem 3.11 and Corollary 3.2.10 of [13] we immediately get the following result.

Corollary 3.12 (cf. with Theorem 1 from [27]). *Let* $M = (r_n v_k) \in F^+$ *be coregular with* $v_k > 0$. *Then* $c_M = c$ *if and only if* $\left(V_n / v_n\right) \in m$.

Further we describe the summability domains of $M \in F^+$ *via* C^1. For doing it, we need some notions and auxiliary results. We remember that matrices *A* and *B* are said to be consistent if $\lim_n B_n x = \lim_n A_n x$ for every $x \in c_A \cap c_B$.

Lemma 3.13 ([13], Theorem 2.6.2). *Let A be a normal matrix and B a triangular matrix. Then B is stronger than and consistent with A if and only if* $C = BA^{-1}$ *is regular.*

Theorem 3.14 (cf. [30], Theorem 2.1). *Let* $M = (r_n v_k) \in F^+$ *be a regular normal* matrix, *where the sequence* (v_k) *is monotone and;*

$m < (n+1) r_n v_n < M$,

for some positive constants m and M. Then $c_M = c_{C^1}$, *and M and* C^1 *are consistent.*

Proof. First we show that $c_{C^1} \subset c_M$. Let $C^{-1} = (c_{nk})$ be the inverse matrix of C^1. Then with the help of (2.1) we obtain;

$$c_{nk} = \begin{cases} n+1 & \text{if } n = k, \\ -(n+1) & \text{if } k = k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = MC^{-1} := (d_{nk})$. Then for $k < n$ we have,

$$d_{nk} = \sum_{j=k}^{n} r_n v_j c_{jk} = (k+1) r_n (v_k - v_{k+1}),$$

$d_{nn} = (n+1) r_n v_n$ and $d_{nk} = 0$ for $k > n$.

Therefore $\lim_n d_{nk} = 0$, since $r = 0$ by Lemma 3.1.

Assume that (v_k) is non-increasing. Then,

$$\sum_{k=0}^{n} |d_{nk}| = \sum_{k=0}^{n} d_{nk} = (n+1) r_n v_n + r_n \sum_{k=0}^{n} (k+1)(v_k - v_{k+1})$$

$$= (n+1) r_n v_n + r_n \left[\sum_{k=0}^{n-1} (k+1) v_k - \sum_{k=0}^{n-1} (k+1) v_{k+1} \right]$$

$$= (n+1) r_n v_n + r_n \left[v_0 - n v_n + \sum_{k=0}^{n-1} ((k+1) v_k - k v_k) \right]$$

$$= r_n v_n + r_n v_0 + r_n \sum_{k=0}^{n-1} v_k = r_n v_0 + r_n V_n .$$

This implies by Lemma 3.1 that *D* is regular, since $q = 1$ and $r = 0$ by the regularity of *M*.

Assume that (v_k) is non-decreasing. Then,

$$\sum_{k=0}^{n} |d_{nk}| = (n+1) r_n v_n + r_n \sum_{k=0}^{n} (k+1)(v_{k+1} - v_k)$$

$$= (n+1) r_n v_n + r_n \left[\sum_{k=0}^{n-1} (k+1) v_{k+1} - \sum_{k=0}^{n-1} (k+1) v_k \right]$$

$$= (n+1)r_n v_n + r_n n v_n - r_n v_0 + r_n \sum_{k=0}^{n-1} (kv_k - (k+1)v_k)$$

$$O(1) + o(1) - r_n \sum_{k=0}^{n-1} v_k = O(1).$$

As,

$$De = MC^{-1}e = Me; e = (1,1,...)$$

and *M* is regular, then $\lim_n D_n e = 1$ by Lemma 3.1. Hence D is regular by Lemma 3.1. Therefore $c_{C^1} \subset c_M$ by Lemma 3.13.

For showing $c_M \subset c_{C^1}$ it is sufficient to prove that D^{-1} is regular. The proof of this statement we refer to [30], p. 589 -591. Thus, by Lemma 3.13, $c_M = c_{C^1}$, and *M* and C^1 are consistent.

4. INCLUSION THEOREMS

In this section we study the transformations of absolute summability domains of normal matrices by factorable matrices. Let;

$$F_v^{cs} := \{ M \in F : (r_n) \in cs \},$$

$$F_v^{l} := \{ M \in F : (r_n) \in l \},$$

for a given sequence $v = (v_k)$.

Lemma 4.1 ([3], p. 405). *Let A be a normal matrix, where* $e^0 = (1,0,0,...) \in l_A$.

(i) *If* $l_A \subset cs_M$ *for* $M \in F$, then $(r_n) \in cs$.

(ii) *If* $l_A \subset l_M$ *for* $M \in F$, then $(r_n) \in l$.

Proof follows from the equality $M_n e^0 \in r_n v_0$.

Theorem 4.2 ([3], Theorem 2.2). *Let* $A = (a_{nk})$ *be a normal matrix and* $A^{-1} = (c_{nk})$ *its inverse matrix. Then* $l_A \subset l_M$ *for every* $M \in F_v^l$ *if and only if;*

$$\sum_{n=l}^{m} v_n c_{nl} = O(1). \tag{4.1}$$

Proof. For every $x = (x_k) \in l_A$ we can write;

$$x_k = \sum_{n=l}^{k} c_{kl} z_l,$$

where $z_l = A_l x$. Therefore for $M \in F$ and for every $x \in l_A$ we obtain;

$$M_n x = r_n L_n(z), \tag{4.2}$$

where,

$$L_n(z) := \sum_{l=0}^{n} \left(\sum_{k=l}^{n} v_k c_{kl} \right) z_l.$$

As *A* is normal, then for every $z = (z_l) \in l$ there exists $x \in l_A$ such that $A_l x = z_l$. This implies by (4.2) that $Mx \in l$ for each $M \in F_v^l$ and each $x \in l_A$ if and only if $(r_n L_n(z)) \in l$ for every $(r_n) \in l.$ The relation $(r_n L_n(z)) \in l$ holds if and only if;

$$L_n(z) = O_z(1) \tag{4.3}$$

for each $z \in l.$ As;

$$L_n(z) = \sum_{l=0}^{n} g_{nl} z_l, \text{ where } g_{nl} = \sum_{k=l}^{n} v_k c_{kl},$$

for every $z \in l,$ then (4.3) holds for every $z \in l$ if and only if $G = (g_{nk})$ is a transform from *l* into *m*. By Proposition 6 of [35], *G* transforms *l* into *m* if and only if condition (4.1) is fulfilled.

Theorem 4.3. *Let* $A = (a_{nk})$ *be a normal matrix and* $A^{-1} = (c_{nk})$ *its inverse matrix. Then* $l_A \subset cs_M$ *for every* $M \in F_v^{cs}$ *if and only if;*

$$\sum_{n=l}^{\infty} \left| v_n c_{nl} \right| = O(1). \tag{4.4}$$

Proof. For the proof we refer to Theorem 2.3 from [3].

We note that (4.1) follows from (4.4). Hence from Theorems 4.2 and 4.3 we obtain immediately the following corollary.

Corollary 4.4. *Let* $A = (a_{nk})$ *be a normal matrix and* $v = (v_k)$ *a sequence of complex numbers. If* $l_A \subset cs_M$ *for every* $M \in F_v^{cs}$*, then* $l_A \subset l_M$ *for every* $M \in F_v^{l}$.

From Theorems 4.2 and 4.3 we also obtain immediately the following result.

Corollary 4.5. *Let* $A=(a_{nk})$ *be a normal matrix and* $v=(v_k)$ *a sequence of complex numbers. If* $l_A \subset cs_M$ *for every* $M \in F_v^{cs}$ *or* $l_A \subset l_M$ *for every* $M \in F_v^l$, *then;*

$$v_n c_{nn} = O(1). \tag{4.5}$$

Now we consider the special case if *A* is the series-to-series Cesàro matrix C^{α}, where $\alpha \in C$ and $\alpha \neq -1,-2,...$; i.e., $C^{\alpha}=(a_{nk})$ is a lower triangular matrix with (see [12], p. 84);

$$a_{nk} = \frac{kA_{n-k}^{\alpha-1}}{nA_n^{\alpha}},$$

for $k \le n$, where $A_n^{\alpha} = \begin{pmatrix} n+\alpha \\ n \end{pmatrix}$ are Cesàro numbers. The inverse matrix $A^{-1}=(c_{nk})$ of C^{α} is the lower triangular matrix with (see [12], p. 86),

$$c_{nk} = \frac{kA_k^{\alpha}A_{n-k}^{-\alpha-1}}{n},$$

for $k \le n$. To prove next results, the following properties of Cesàro numbers are necessary (see [12], p. 77-81):

$$A_0^{-1}=1;\ A_n^{-1}=0 \text{ for } n \ge 1, \tag{4.6}$$

$$\left|A_n^{\alpha}\right| \le K_1(n+1)^{\operatorname{Re}\alpha} \text{ for } \alpha \in C, K_1>0, \tag{4.7}$$

$$\left|A_n^{\alpha}\right| \ge K_2(n+1)^{\operatorname{Re}\alpha} \text{ for } \alpha \in C, \alpha \neq -1,-2,...; K_2>0. \tag{4.8}$$

We see that $e^0 \in l_{C^{\alpha}}$, since $C_n^{\alpha}e^0 = a_{n0}$ and $a_{00}=1$, $a_{n0}=0$ for $n \ge 1$. Hence from Lemma 4.1 we immediately obtain:

Corollary 4.6. *Let* $\alpha \in C, \alpha \neq -1,-2,....$

(i) *If* $l_{C^{\alpha}} \subset cs_M$ *for* $M \in F$, then $(r_n) \in cs$.

(ii) *If* $l_{C^{\alpha}} \subset l_M$ *for* $M \in F$, then $(r_n) \in l$.

Using Theorem 4.2 we prove the following statement.

Proposition 4.7 ([3], Proposition 3.2). *Let* $\alpha \in C$ *with* $\operatorname{Re}\alpha > 0$ or $\alpha = 0$, and $v=(v_k)$ *be defined by* $v_k = 1/A_k^t$, $t \in C$. Then $l_{C^{\alpha}} \subset cs_M$ *for every* $M \in F_v^{cs}$ if and only if $\operatorname{Re}\alpha \le \operatorname{Re}t$.

Proof. Condition (4.4) we can rewrite as follows:

$$T_l := lA_l^{\alpha}\sum_{n=l}^{\infty}\left|\frac{A_{n-l}^{-\alpha-1}}{nA_n^t}\right| = O(1). \tag{4.9}$$

Since $A_l^0 = 1$, then (4.8) for $\alpha = 0$ is equivalent to the condition;

$$\frac{1}{A_k^t} = O(1) \tag{4.10}$$

by (4.6). Conditions (4.7) and (4.8) imply that (4.10) is fulfilled if and only if $\operatorname{Re}t \ge 0$.

Let now $\operatorname{Re}\alpha > 0$. Using (4.6) we obtain that (4.5) can be presented as;

$$\left|\frac{A_l^{\alpha}}{A_l^t}\right| = O(1). \tag{4.11}$$

Condition (4.11) holds by (4.7) and (4.8) if and only if $\operatorname{Re}\alpha \le \operatorname{Re}t$. With the help of (4.7) and (4.8) we get for $\operatorname{Re}\alpha \le \operatorname{Re}t$ that;

$$T_l = O(1)(l+1)^{\operatorname{Re}\alpha+1}\sum_{n=l}^{\infty}\frac{(n-l+1)^{-\operatorname{Re}\alpha-1}}{(n+l+1)^{\operatorname{Re}t+1}}$$

$$= O(1)(l+1)^{\operatorname{Re}\alpha+1}\sum_{n=0}^{\infty}\frac{1}{(n+1)^{\operatorname{Re}\alpha+1}(n+l+1)^{\operatorname{Re}t+1}}$$

$$= O(1)(l+1)^{\operatorname{Re}(\alpha-t)}\sum_{n=0}^{\infty}\frac{1}{(n+1)^{\operatorname{Re}\alpha+1}\left(\frac{n}{l+1}+1\right)^{\operatorname{Re}t+1}}$$

$$= O(1)\sum_{n=0}^{\infty}\frac{1}{(n+1)^{\operatorname{Re}\alpha+1}} = O(1).$$

Thus, $l_{C^{\alpha}} \subset cs_M$ by Theorem 4.2.

Proposition 4.8. *Let* $\alpha \in C$ *with* $\operatorname{Re}\alpha > 0$ or $\alpha = 0$, and $v=(v_k)$ *be defined by* $v_k = 1/A_k^t$, $t \in C$. Then $l_{C^{\alpha}} \subset l_M$ *for every* $M \in F_v^l$ if and only if $\operatorname{Re}\alpha \le \operatorname{Re}t$.

Proof. For the proof we refer to Proposition 3.3 from [3].

5. MATRIX TRANSFORMS FROM c_A INTO c_B BY FACTORABLE MATRICES

In this section we consider matrix transforms from c_A into c_B for certain matrices *A* and *B*.

Proposition 5.1 ([4], Proposition 3.1). *Let* $A=(a_{nk})$ *be a matrix with* $e^0 \in c_A$, $B=(b_{nk})$ *an arbitrary matrix with real or complex entries and* $M \in F$. If $M \in (c_A, c_B)$, *then* $(r_n) \in c_B$.

Proof follows from the equality $M_n e^0 = r_n v_0$.

Theorem 5.2 ([4], Theorem 3.2). *Let* $A=(a_{nk})$, $B=(b_{nk})$ *be matrices with real or complex entries,* $M \in F$ *and* $B^t=(b_{sn}^t)$ *a matrix defined by the relation* $b_{sn}^t = b_{sn} r_n$. *Then* $M \in (c_A, c_B)$ *if;*

$$(v_k x_k) \in cs \text{ for each } x \in c_A, \tag{5.1}$$

$$B^t \text{ is conservative.} \tag{5.2}$$

Proof follows from the relation;

$$\sum_n b_{sn} M_n x = \sum_n b_{sn}^t \sum_{k=0}^n v_k x_k \ ,$$

for each $x \in c_A$.

We say that a matrix *A* is series-to-sequence conservative (shortly, Sr-Sq conservative) if $Ax \in c$ for every $x \in cs$, and series-to-sequence regular (shortly, Sr-Sq regular) if;

$$\lim_n A_n x = \lim_n \sum_{k=0}^n x_k \ ,$$

for every $x \in cs$.

Proposition 5.3 ([4], Proposition 3.3). *Let* $B=(b_{nk})$ *be Sr-Sq regular, where* $b_{nk}>0$ for all *n* and *k*, and (r_n) a sequence of complex numbers. Then condition (5.2) holds *if and only if* $(r_n) \in l$.

Proof. Necessity. Let B^t is conservative. Then by Lemma 3.1,

$$T_s := \sum_n |b_{sn} r_n| = \sum_n b_{sn} |r_n| = O(1). \tag{5.3}$$

If $(r_n) \notin l$, then (see [14], p. 92) $\lim_s T_s = \infty$; i.e., relation (5.3) does not hold. Thus $(r_n) \in l$.

Sufficiency. Assume that $(r_n) \in l$. We prove that all conditions of Lemma 3.1 hold for $A = B^t$. From the Sr-Sq regularity of *B* we obtain that $(r_n) \in c_B$, $b_{nk} = O(1)$, and there exist the finite limits $\lim_n b_{nk}$ by Proposition 17 of [35]. Hence

$$T_s = O(1) \sum_n |r_n| = O(1).$$

Consequently all conditions of Lemma 3.1 are fulfilled for $A = B^t$; i.e., condition (5.2) holds *by* Lemma 3.1.

Theorem 5.4 ([4], Theorem 3.4). *Let* $A=(a_{nk})$, $B=(b_{nk})$ *be matrices with real or complex entries,* $l \in c_B$, $(r_n) \in l$ and $M \in F$. *Then* $M \in (c_A, c_B)$ *if condition (5.1) holds.*

Proof. For each $x \in c_A$ we denote;

$$S_n := \sum_{k=0}^n v_k x_k .$$

Since it follows from (5.1) that $(S_n) \in c$ for each $x \in c_A$, then (S_n) is bounded for each $x \in c_A$. This implies;

$$\sum_n |M_n x| = \sum_n |r_n S_n| = O(1) \sum_n |r_n| = O(1)$$

for every $x \in c_A$. Hence, due to $l \in c_B$, $M \in (c_A, c_B)$.

Now we consider the special case if *A* is the series-to-sequence Cesàro matrix C^α, where $\alpha \in C$ and $\alpha \neq -1, -2, \ldots$; i.e., $C^\alpha = (a_{nk})$ is a lower triangular matrix with (see [12], p. 76);

$$a_{nk} = \frac{A_{n-k}^\alpha}{n A_n^\alpha}\ , \ k \le n.$$

Lemma 5.5 ([12], p. 192). *Let* $\alpha \in C$ *with* $\operatorname{Re}\alpha > 0$ or $\alpha = 0$, and $v=(v_k)$ *is a sequence of complex numbers. Then* $(v_k x_k) \in cs$ *for every* $(x_k) \in c_{C^\alpha}$ *if and only if;*

$$v_k = O[(k+1)^{-\operatorname{Re}\alpha}], \tag{5.4}$$

$$\sum_k (k+1)^{\operatorname{Re}\alpha} \left|\Delta_k^{\alpha+1} v_k\right| = O(1), \qquad (5.5)$$

where,

$$\Delta_k^{\alpha+1} v_k := \sum_{n=k}^{\infty} A_{n-k}^{-\alpha-2} v_n.$$

Further we also need the relation (see [12], p. 81)

$$\sum_{n=k}^{\infty} \frac{A_{n-k}^{\alpha}}{A_n^{\beta}} = \frac{\beta}{\beta-\alpha-1} \cdot \frac{1}{A_k^{\beta-\alpha-1}}$$

for $\operatorname{Re}\beta \geq 0,\ \operatorname{Re}(\beta-\alpha) > 1, k = 1, 2, \ldots.$ (5.6)

Proposition 5.6 ([4], Theorem 4.1). *Let* $\alpha \in C$ *with* $\operatorname{Re}\alpha > 0$ or $\alpha = 0$ and $B = (b_{nk})$ *be a matrix with* $l \in c_B$. *Let* $M \in F$ *with* $v_k := 1/A_k^t$, $t \in C, \operatorname{Re} t > 0$ and $(r_n) \in l$. Then $M \in (c_{C^\alpha}, c_B)$ if $\operatorname{Re}\alpha \leq \operatorname{Re} t$.

Proof. It is sufficient to show by Theorem 5.4 that (5.1) is satisfied for $A = C^\alpha$ and $v_k = 1/A_k^t$. Using (4.8) and (5.6) we obtain;

$$\sum_k (k+1)^{\operatorname{Re}\alpha} \left|\Delta_k^{\alpha+1} v_k\right| = \sum_k (k+1)^{\operatorname{Re}\alpha} \left|\sum_{n=k}^{\infty} \frac{A_{n-k}^{-\alpha-2}}{A_n^t}\right|$$

$$= \sum_k (k+1)^{\operatorname{Re}\alpha} \left|\frac{t}{t+\alpha+1} \cdot \frac{1}{A_k^{t+\alpha+1}}\right|$$

$$= O(1) \sum_k \frac{(k+1)^{\operatorname{Re}\alpha}}{(k+1)^{\operatorname{Re}(t+\alpha)+1}} = O(1) \sum_k \frac{1}{(k+1)^{\operatorname{Re}(t+1)}} = O(1);$$

i.e., condition (5.5) holds. Condition (5.4) also holds, because by (4.8) there exists $K > 0$, such that;

$$\left|\frac{1}{A_k^t}\right| \leq \frac{1}{K(k+1)^{\operatorname{Re} t}} = O(1)(k+1)^{-\operatorname{Re} t} = O(1)(k+1)^{-\operatorname{Re}\alpha}.$$

Hence condition (5.1) holds by Lemma 5.5. Thus $M \in (c_{C^\alpha}, c_B)$ by Theorem 5.4.

Proposition 5.7. *Let* $\alpha \in C$ *with* $\operatorname{Re}\alpha > 0$ or $\alpha = 0$ and $B = (b_{nk})$ *be a matrix with* $l \in c_B$. *Let* $M \in F$ *with* $v_k := y^k$, $y \in C$ and $(r_n) \in l$. Then $M \in (c_{C^\alpha}, c_B)$ if $|y| < 1$.

Proof. For the proof we refer to Theorem 4.2 from [4].

REFERENCES

[1] Aasma A, Dutta H and Natarajan PN. An Introductory Course in Summability Theory. JOHN WILEY & SONS: Hoboken 2017. https://doi.org/10.1002/9781119397786

[2] Aasma A. Matrix Transforms of Summability Domains of Normal Series-to-Series Matrices. J Adv Appl Comput Math 2014; 1: 35-39. https://doi.org/10.15377/2409-5761.2014.01.02.1

[3] Aasma A. Some inclusion theorems for absolute summability. Appl Math Lett 2012; 25(3): 404-407. https://doi.org/10.1016/j.aml.2011.09.023

[4] Aasma A. Factorable matrix transforms of summability domains of Cesàro matrices. Int J Contemp Math Sci 2011; 6(44): 2201-2206.

[5] Aasma A. On the matrix transformations of absolute summability fields of reversible matrices. Acta Math. Hungar 1994; 64(2): 143-150. https://doi.org/10.1007/BF01874118

[6] Aasma A. Matrix transformations of summability fields of regular perfect matrix methods. Acta et Comment Univ Tartuensis 1994; 970: 3-12.

[7] Aasma A. Transformations of summability fields. Acta et Comment Univ. Tartuensis 1987; 770: 38-51 (in Russian).

[8] Akgun FA and Rhoades BE. Characterizations of H-J Matrices. Filomat 2016; 30 (3): 675-679. https://doi.org/10.2298/FIL1603675A

[9] Akgun FA and Rhoades BE. Factorable generalized Hausdorff matrices. J Advanced Math Studies 2010; 3(1): 1-8.

[10] Alp a' r L. On the linear transformations of series summable in the sense of Ces a' ro. Acta Math Hungar 1982; 39(1): 233-243. https://doi.org/10.1007/BF01895236

[11] Alp a' r L. Sur certainschangements de variable des series de Faber. Studia Sci Math Hungar 1978; 13(1-2): 173-180.

[12] Baron S. Introduction to the theory of summability of series. Valgus: Tallinn 1977 (in Russian).

[13] Boos J. Classical and modern methods in summability. Oxford University Press: Oxford 2000.

[14] Cooke RG. Infinite Matrices and Sequence Spaces. State Publishing Hous of Physics-Mathematics Literature: Moscow 1960 (in Russian).

[15] Hardy GH. Divergent series. Oxford University Press: U.K. 1949.

[16] Rhaly JrHC. Posinormality, coposinormality, and supraposinormality for some triangular operators. Sarajevo J Math 2016; 12(24)(1): 125-140.

[17] Rhaly JrHC. Concerning the Cesàro matrix and its immediate offspring. Acta Math Univ Comenian (N.S.) 2015; 84(1): 27-38.

[18] Rhaly JrHC. A superclass of the posinormal operators. New York J Math 2014; 20: 497-506.

[19] Rhaly Jr.HC and Rhoades BE. The weighted mean operator on l^2 with the weight sequence $w_n = n+1$ is hyponormal. New Zealand J Math 2014; 44: 103-106.

[20] Rhaly Jr.HC and Rhoades BE. Posinormal factorable matrices with a constant main diagonal. Rev Un Mat Argentina 2014; 55(1): 19-24.

[21] Rhaly Jr.HC. A comment on coposinormal operators. Matematiche (Catania) 2013; 68 (1): 83-86.

[22] Rhaly Jr.H.C and Rhoades BE. Conditions for factorable matrices to be hyponormal and dominant. Sib Elektron Mat Izv 2012; 9: 261-265.

[23] Rhaly Jr.H.C. Posinormal factorable matrices whose interrupter is diagonal. Mathematica 2011; 53(76) (2): 181-188.

[24] Rhaly Jr.H.C. Discrete generalized Cesàro operators. Proc Amer Math So 1986; 34: 225-232.

[25] Rhaly Jr.H.C. p- Cesàro matrices, Houston J Math 1989; 15(1): 137-146.

[26] Rhaly Jr.H.C. Terraced matrices. Bull London Math Soc 1989; 21(4): 399-406. https://doi.org/10.1112/blms/21.4.399

[27] Rhoades BE. An extension of two results of Hardy. Sarajevo J Math 2013; 9 (21): 95-100. https://doi.org/10.5644/SJM.09.1.08

[28] Rhoades BE and Yildirim M. The spectra and fine spectra of factorable matrices on c_0. Math Commun 2011; 16(1): 265-270.

[29] Rhoades BE and Sen P. Lower bounds for some factorable matrices. Int J Math Math Sci 2006: Art. ID 76135: 1-13.

[30] Rhoades BE. Some sequence spaces which include c_0 and c. Hokkaido Math J 2006; 35: 587-599. https://doi.org/10.14492/hokmj/1285766418

[31] Rhoades BE and Yildirim M. Spectra for Factorable Matrices on l^p. Integr equ oper Theory 2006; 55: 111-126.

[32] Rhoades BE. Lower bounds for some matrices. Linear and Multilinear Algebra 1987; 20 (4): 47-352. https://doi.org/10.1080/03081088708817767

[33] Rhoades BE. Lower bounds for some matrices II. Linear and Multilinear Algebra 1990; 26(1-2): 49-58. https://doi.org/10.1080/03081089008817965

[34] Sinnamon G. Masked factorable matrices. Proc Royal Soc Edinburgh 2002; 132A: 245-254. https://doi.org/10.1017/S0308210500001608

[35] Stieglitz M and Tietz H. Matrix transformationen von Folgenräumen: eine Ergebnis $\ddot{u}$ bersicht. Math Z 1977; 154: 1-16. https://doi.org/10.1007/BF01215107

[36] Thorpe B. Matrix transformations of Ces a' ro summable series. Acta Math Hungar 1986; 48(3-4): 255-265. https://doi.org/10.1007/BF01951350

[37] Wilansky A. Functional Analysis. Blaisdell Publ Co: New York 1964.

[38] Zeltser M. Factorable matrices and their associated Riesz matrices. Proc Estonian Acad Sci 2014; 63(4): 379-386. https://doi.org/10.3176/proc.2014.4.03

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