Other Subordination Results for Fractional Integral Associated with Dziok-Srivastava Operator

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Abstract: In this paper we have discussed differential subordination properties associated with the fractional integral by using Dziok-Srivastava operator.

Keywords: Analytic function, differential subordination, fractional integral, Dziok-Srivastava operator.

1. INTRODUCTION

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $H(U)$ the space of holomorphic functions in U.

Let $A_n=\{f\in H(U):f(z)=z+a_{n+1}z^{n+1}+\ldots,\in U\}$ with $A_1=A$ and $H[a,n]=\{f\in H(U):f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots,z\in U\}$ for $a\in\mathbb{C}$ and $n\in\mathbb{N}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$, if there is a function w analytic in U, with $w(0)=0$, $|w(z)|<1$, for all $z\in U$, such that $f(z)=g(w(z))$ for all $z\in U$. If g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subseteq g(U)$.

Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ and h an univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z),zp'(z),z^2p''(z);z)\prec h(z),\qquad z\in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p\prec q$ for all p satisfying (1.1).

A dominant $\tilde{q}$ that satisfies $\tilde{q}\prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Definition 1.1 ([3]) For $f\in A$, the Dziok-Srivastava operator is defined by

$$H_m^l(\alpha_1,\alpha_2,...,\alpha_l;\beta_1,\beta_2,...,\beta_m):A\to A,$$

$$H_m^l(\alpha_1,\alpha_2,...,\alpha_l;\beta_1,\beta_2,...,\beta_m)f(z)=$$
$$z+\sum_{j=2}^{\infty}\frac{(\alpha_1)_{j-1}(\alpha_2)_{j-1}...(\alpha_l)_{j-1}}{(\beta_1)_{j-1}(\beta_2)_{j-1}...(\beta_m)_{j-1}(j-1)!}a_jz^j, \tag{1.2}$$

$\alpha_i\in\mathbb{C}$, $i=1,2,...,l$, $\beta_k\in\mathbb{C}\setminus\{0,-1,-2,...\}$, $k=1,2,...,m$,

where $(x)_j$ is the Pochhammer symbol defined, in terms of the Gamma function by

$$(x)_j=\frac{\Gamma(x+j)}{\Gamma(x)}=\begin{cases}1,\text{ if } j=0\text{ and }x\in\mathbb{C}\setminus\{0\},\\ x(x+1)...(x+j-1),\text{ if }j\in\mathbb{N}\text{ and }x\in\mathbb{C}.\end{cases}$$

For simplicity, we write

$$H_m^l[\alpha_1]f(z)=H_m^l(\alpha_1,\alpha_2,...,\alpha_l;\beta_1,\beta_2,...,\beta_m)f(z). \tag{1.3}$$

Definition 1.2 ([1]) The fractional integral of order λ ($\lambda>0$) is defined for a function f by

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z\frac{f(t)}{(z-t)^{1-\lambda}}dt, \tag{1.4}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t)>0$.

From Definitions 1.1 and 1.2, we get ([4])

$$D_z^{-\lambda}H_m^l[\alpha_1]f(z)=\frac{1}{\Gamma(2+\lambda)}z^{1+\lambda}+$$
$$\sum_{j=2}^{\infty}\frac{\Gamma(j+1)}{\Gamma(j+1+\lambda)(j-1)!}\frac{(\alpha_1)_{j-1}(\alpha_2)_{j-1}...(\alpha_l)_{j-1}}{(\beta_1)_{j-1}(\beta_2)_{j-1}...(\beta_m)_{j-1}(j-1)!}a_jz^{j+\lambda}. \tag{1.5}$$

From [4] we need this result

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$$z\left(D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)' = \alpha_1 D_z^{-\lambda}H_m^l\left[\alpha_1+1\right] f\left(z\right)-\left[\alpha_1-\left(1+\lambda\right)\right]D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right). \quad (1.6)$$

Lemma 1.1 (Miller and Mocanu [2]) Let g be a convex function in U and let $h(z)=g(z)+n\alpha zg'(z)$, for $z\in U$, where $\alpha>0$ and n is a positive integer.

If $p(z)=g(0)+p_nz^n+p_{n+1}z^{n+1}+\ldots$, $z\in U$, is holomorphic in U and

$$p(z)+\alpha zp'(z)\prec h(z), \quad z\in U,$$

then

$$p(z)\prec g(z), \quad z\in U,$$

and this result is sharp.

2. MAIN RESULTS

Theorem 2.1 Let g be a convex function, $g(0)=0$ and let h be the function $h(z)=g(z)+\lambda zg'(z)$, for $z\in U$.

If $f\in A$ and satisfies the differential subordination

$$\left(D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)'\prec h(z), \quad \text{for } z\in U, \quad (2.1)$$

then

$$\frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}\prec g(z), \quad \text{for } z\in U,$$

and this result is sharp.

Proof. Consider $p(z)=\frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}$, for $z\in U$.

Let $D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)=zp(z)$, for $z\in U$. Differentiating we obtain $\left(D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)' = p(z)+zp'(z)$, for $z\in U$.

Then (2.1) becomes

$$p(z)+zp'(z)\prec h(z)=g(z)+\lambda zg'(z), \quad \text{for } z\in U.$$

By using Lemma 1.1, we have

$$p(z)\prec g(z), \quad \text{for } z\in U, \quad i.e. \quad \frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}\prec g(z),$$

for $z\in U$.

Theorem 2.2 Let g be a convex function, $g(0)=0$ and let h be the function $h(z)=g(z)+\lambda zg'(z)$, $z\in U$.

If $f\in A$, $\delta>0$, and satisfies the differential subordination

$$\left(\frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}\right)^{\delta-1}\left(D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)'\prec h(z), z\in U, \quad (2.2)$$

then

$$\left(\frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}\right)^{\delta}\prec g(z), \quad z\in U,$$

and this result is sharp.

Proof. Consider $p(z)=\left(\frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}\right)^{\delta}$, $z\in U$.

Differentiating we obtain $\left(\frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}\right)^{\delta-1}$ $\left(D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)' = p(z)+\frac{1}{\delta}zp'(z)$, $z\in U$.

Then (2.2) becomes

$$p(z)+\frac{1}{\delta}zp'(z)\prec h(z)=g(z)+\lambda zg'(z), \quad z\in U.$$

By using Lemma 1.1, we have

$$p(z)\prec g(z), z\in U, \text{i.e.} \left(\frac{D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)}{z}\right)^{\delta}\prec g(z), z\in U.$$

Theorem 2.3 Let g be a convex function such that $g(0)=\frac{1}{1+\lambda}$ and let h be the function $h(z)=g(z)+zg'(z)$, $z\in U$.

If $f\in A$ and the differential subordination

$$\frac{\alpha_1^2\left(D_z^{-\lambda}H_m^l\left[\alpha_1+1\right]f\left(z\right)\right)^2-\alpha_1\left(\alpha_1+1\right)D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\cdot D_z^{-\lambda}H_m^l\left[\alpha_1+2\right]f\left(z\right)}{\left(\alpha_1D_z^{-\lambda}H_m^l\left[\alpha_1+1\right]f\left(z\right)-\left[\alpha_1-\left(1+\lambda\right)\right]D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)^2}+$$

$$\frac{2\alpha_1D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\cdot D_z^{-\lambda}H_m^l\left[\alpha_1+1\right]f\left(z\right)-\left[\alpha_1-\left(1+\lambda\right)\right]\left(D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)^2}{\left(\alpha_1D_z^{-\lambda}H_m^l\left[\alpha_1+1\right]f\left(z\right)-\left[\alpha_1-\left(1+\lambda\right)\right]D_z^{-\lambda}H_m^l\left[\alpha_1\right]f\left(z\right)\right)^2}\prec h(z), \quad z\in U \quad (2.3)$$

holds, then

$$\frac{D_z^{-\lambda}H_m^l[\alpha_1]f(z)}{z\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)'} \prec g(z), \quad z \in U.$$

This result is sharp.

Proof. Let $p(z) = \dfrac{D_z^{-\lambda}H_m^l[\alpha_1]f(z)}{z\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)'}$.

Differentiating, we obtain

$$1 - \frac{D_z^{-\lambda}H_m^l[\alpha_1]f(z)\cdot\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)''}{\left[\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)'\right]^2} = p(z) + zp'(z),$$

$z \in U$.

After a short calculation, using relation (1.6) we obtain

$$1 - \frac{D_z^{-\lambda}H_m^l[\alpha_1]f(z)\cdot\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)''}{\left[\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)'\right]^2} =$$

$$\frac{\alpha_1^2\left(D_z^{-\lambda}H_m^l[\alpha_1+1]f(z)\right)^2 - \alpha_1(\alpha_1+1)D_z^{-\lambda}H_m^l[\alpha_1]f(z)\cdot D_z^{-\lambda}H_m^l[\alpha_1+2]f(z)}{\left(\alpha_1 D_z^{-\lambda}H_m^l[\alpha_1+1]f(z) - [\alpha_1-(1+\lambda)]D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)^2} +$$

$$\frac{2\alpha_1 D_z^{-\lambda}H_m^l[\alpha_1]f(z)\cdot D_z^{-\lambda}H_m^l[\alpha_1+1]f(z) - [\alpha_1-(1+\lambda)]\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)^2}{\left(\alpha_1 D_z^{-\lambda}H_m^l[\alpha_1+1]f(z) - [\alpha_1-(1+\lambda)]D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)^2}$$

Using the notation in (2.3), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 1.1, we have

$$p(z) \prec g(z), z \in U, \text{ i.e. } \frac{D_z^{-\lambda}H_m^l[\alpha_1]f(z)}{z\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)'} \prec g(z), \quad z \in U,$$

and this result is sharp.

Theorem 2.4 Let g be a convex function such that $g(0) = 0$ and let h be the function $h(z) = g(z) + \dfrac{\lambda}{\alpha_1 - \lambda} zg'(z)$, for $z \in U$.

If $f \in A$ and the differential subordination

$$\frac{\alpha_1(\alpha_1+1)}{\alpha_1-\lambda}\frac{D_z^{-\lambda}H_m^l[\alpha_1+2]f(z)}{z} - \alpha_1\frac{D_z^{-\lambda}H_m^l[\alpha_1+1]f(z)}{z} \prec h(z), \quad \text{for } z \in U \tag{2.4}$$

holds, then

$$\left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)' \prec g(z), \quad \text{for } z \in U.$$

This result is sharp.

Proof. Let

$$p(z) = \left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)' \tag{2.5}$$

Differentiating and using relation (1.6), we obtain

$$\frac{\alpha_1(\alpha_1+1)}{\alpha_1-\lambda}\frac{D_z^{-\lambda}H_m^l[\alpha_1+2]f(z)}{z} - \alpha_1\frac{D_z^{-\lambda}H_m^l[\alpha_1+1]f(z)}{z} = p(z) + \frac{1}{\alpha_1-\lambda}zp'(z).$$

Using the notation in (2.5), the differential subordination becomes

$$p(z) + \frac{1}{\alpha_1-\lambda}zp'(z) \prec h(z) = g(z) + \frac{\lambda}{\alpha_1-\lambda}zg'(z).$$

By using Lemma 1.1, we have

$$p(z) \prec g(z), \quad \text{for } z \in U, \quad i.e. \quad \left(D_z^{-\lambda}H_m^l[\alpha_1]f(z)\right)' \prec g(z),$$
for $z \in U$,

and this result is sharp.

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