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# Vectorial Prabhakar HardyType Generalized Fractional Inequalities under Convexity

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### ABSTRACT

We present a detailed great variety of Hardy type fractional inequalities under convexity and Lp norm in the setting of generalized Prabhakar and Hilfer fractional calculi of left and right integrals and derivatives. The radial multivariate case of the above over a spherical shell is developed in detail to all directions. Many inequalities are of vectorial splitting rational  $Lp$  type or of separating rational  $Lp$  type, others involve ratios of functions and of fractional integral operators.

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### 1. Background

This work is inspired by [3-11].

Here we consider the Prabhakar function (also known as the three parameter Mittag-Laffler function), (see [6], p. 97; [5])

 ! = =0 , k k k z k k E z (1)

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as the three parameter Mittag-Laffler function), (see [6],<br>  $\frac{(\gamma)_k}{(\alpha k + \beta)} z^k$ ,<br>  $\beta > 0, z \in \mathbb{R}$ , and  $(\gamma)_k = \gamma(\gamma + 1)...(\gamma + k - 1)$ . It is where  $\Gamma$  is the gamma function;  $\alpha, \beta, \gamma \in \mathbb{R}: \alpha, \beta > 0, z \in \mathbb{R}$ , and  $(\gamma)_k = \gamma(\gamma + 1)...(\gamma + k - 1)$ . It is ial Prabhakar Hardy Type Generalized Fractional Inequalities<br> **.ackground**<br>
is work is inspired by [3-11].<br>
ere we consider the Prabhakar function (also known as the three parameter Mit<br>
[5])<br>  $E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \$  $\frac{0}{a}$ <sub>a</sub> $(z) = \frac{1}{z}$  $E^0_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)}$ d by [3-11].<br>
he Prabhakar function (also known as the three parameter Mittag-Laffler function), (see [6],<br>  $E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{\binom{\gamma_k}{k}}{k! \Gamma(\alpha k + \beta)} z^k$ , (1)<br>
gamma function;  $\alpha, \beta, \gamma \in \mathbb{R} : \alpha, \beta > 0, z \in \mathbb{R}$  $E'_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\langle y \rangle_k}{k! \Gamma(\alpha k + \beta)} z^k,$  (1)<br>
gamma function;  $\alpha, \beta, \gamma \in \mathbb{R} : \alpha, \beta > 0, z \in \mathbb{R}$ , and  $\langle y \rangle_k = \gamma(y+1)...(y+k-1)$ . It is<br>
and  $x \in [a, b]$ ;  $f \in C([a, b])$  Let also  $\psi \in C^1([a, b])$  which is increasing. The

Here we follow [4].

Let  $a,b\in{\sf R}$  ,  $a\leq b$  and  $x\in[a,b];\; f\in C([a,b]).$  Let also  $\psi\in C^1([a,b])$  which is increasing. The left and right Prabhakar fractional integrals with respect to  $\psi$  are defined as follows:

$$
\left(e^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x) = \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\mu-1} E^{\gamma}_{\rho,\mu} \left[\omega(\psi(x)-\psi(t))^{\rho}\right] f(t)dt, \tag{2}
$$

and

$$
\left(e^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x) = \int_x^b \psi'(t)\left(\psi(t) - \psi(x)\right)^{\mu-1} E_{\rho,\mu}^{\gamma}\left[\omega(\psi(t) - \psi(x))^{\rho}\right]f(t)dt, \tag{3}
$$

where  $\rho, \mu > 0$ ;  $\gamma, \omega \in \mathsf{R}$ .

Functions (2) and (3) are continuous ([4]).

Next, additionally, assume that  $\psi^{'}(x) \neq 0$  over  $[a,b]$  and let  $\psi, f \in C^\infty([a,b])$ , where  $N = \lceil \mu \rceil,$  ( $\lceil \cdot \rceil$  is the ceiling of the number),  $0 \leq \mu \notin N$ . We define the  $\psi$  -Prabhakar-Caputo left and right fractional derivatives of order  $\mu$  ([4]) as follows ( $x \in [a,b]$ ): actional integrals with respect to  $\psi$  are defined as follows:<br>  $(e_{\rho,\mu,\omega,\sigma}^{\vee},f)x) = \int_{a}^{x}\psi'(t)(\psi(x)-\psi(t))^{\omega-1}E_{\rho,\mu}^{\vee}[\omega(\psi(x)-\psi(t))^{\rho}\int f(t)dt,$  (2)<br>  $(e_{\rho,\mu,\omega,\sigma}^{\vee},f)x) = \int_{a}^{b}\psi'(t)(\psi(t)-\psi(x))^{\omega-1}E_{\rho,\mu}^{\vee}[\omega(\psi(t)-\psi(x))^{\rho}\int$ cional integrals with respect to  $\psi$  are defined as follows:<br>  $(e_{\rho,\mu,\omega,\alpha}^{\vee\vee\prime}f)(x) = \int_{a}^{x}\psi'(t)(\psi(x)-\psi(t))^{\omega-1}E_{\rho,\mu}^{\vee}[\omega(\psi(x)-\psi(t))^{\rho}]f(t)dt,$  (2)<br>  $(e_{\rho,\mu,\omega,\alpha}^{\vee\prime\prime}f)(x) = \int_{a}^{b}\psi'(t)(\psi(t)-\psi(x))^{\omega-1}E_{\rho,\mu}^{\vee}[\omega(\psi(t) (e_{\rho,\mu,\omega,\delta-}^{\gamma w}f)x$  =  $\int_{\epsilon}^{\epsilon} \psi'(t)(\psi(t)-\psi(x))^{\rho-1}E_{\rho,\mu}^{\gamma}[\omega(\psi(t)-\psi(x))^{\rho}]f(t)dt$ ,<br>  $\mu > 0; \gamma, \omega \in \mathbb{R}$ .<br>
ons (2) and (3) are continuous ([4]).<br>
additionally, assume that  $\psi'(x) \neq 0$  over [a,b] and let  $\psi, f \in C^{\infty}([a$  $(e_{\rho,n,\omega,b}^{\prime,v}f)x$  =  $\int_{x}^{b}\psi'(t)(\psi(t)-\psi(x))^{n-1}E_{\rho,n}^{\prime}[\omega(\psi(t)-\psi(x))^{v}]f(t)dt$ ,<br>  $u > 0$ ;  $\gamma, \omega \in \mathbb{R}$ .<br>
ns (2) and (3) are continuous ([4]).<br>
assume that  $\psi'(x) \neq 0$  over [a, b] and let  $\psi, f \in C^{\infty}([a,b])$ , where  $N = [\mu]$ ,

$$
\left({}^c D^{\gamma;\psi}_{\rho,\mu,\omega,a+} f\right)(x) = \int_a^x \psi'(t) \big(\psi(x) - \psi(t)\big)^{N-\mu-1} \qquad E^{-\gamma}_{\rho,N-\mu} \Big[\omega(\psi(x) - \psi(t))^{\rho}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^N f(t)dt, \tag{4}
$$

and

$$
\left({}^{c}D_{\rho,\mu,\omega,b-}^{\gamma;\psi}f\right)(x) = (-1)^{N}\int_{x}^{b}\psi'(t)(\psi(t)-\psi(x))^{N-\mu-1} \qquad E_{\rho,N-\mu}^{-\gamma}\Big[\omega(\psi(t)-\psi(x))^{\rho}\Big(\frac{1}{\psi'(t)}\frac{d}{dt}\Big)^{N}f(t)dt. \tag{5}
$$

One can write these (see (4), (5)) as

$$
\left(\begin{matrix} C\ D^{\gamma;\nu}_{\rho,\mu,\omega,a+}f\end{matrix}\right)\!\!\left(x\right) = \left(\begin{matrix} e^{-\gamma;\nu}_{\rho,N-\mu,\omega,a+}f^{[N]}_{\nu}\end{matrix}\right)\!\!\left(x\right),\tag{6}
$$

and

$$
\left(\begin{array}{c}\nC\ D_{\rho,\mu,\omega,b-}^{\gamma;\psi}f(x)\n\end{array}\right) = (-1)^N \left(e_{\rho,N-\mu,\omega,b-}^{-\gamma;\psi}f_{\psi}^{[N]}\right)(x),\tag{7}
$$

where

Journal of Advances in Applied & Computational Mathematics, 8, 2021\n
$$
{}^{C}D_{\rho,\mu,\omega,b-}^{\gamma;\psi}f(x) = (-1)^{N} \left(e_{\rho,N-\mu,\omega,b-}^{-\gamma;\psi}f_{\psi}^{[N]}\right)(x),\tag{7}
$$
\n
$$
f_{\psi}^{[N]}(x) = f_{\psi}^{(N)}f(x) := \left(\frac{1}{\psi^{'}(x)}\frac{d}{dx}\right)^{N}f(x),\tag{8}
$$
\n
$$
\text{ous on } [a,b].
$$

 $\forall x \in [a,b].$ 

Functions (6) and (7) are continuous on  $\big[a,b\big].$ 

Next we define the  $\psi$ -Prabhakar-Riemann Liouville left and right fractional derivatives of order  $\mu$  ([4]) as follows ( $x \in [a,b]$ ,  $f \in C([a,b])$ ):

partial of Advances in Applied & Computational Mathematics, 8, 2021\n
$$
\left(\frac{c}{D_{\rho,\mu,\omega,b-}^{x,\psi}}f(x)\right) = (-1)^{N}\left(e_{\rho,N-\mu,\omega,b-}^{x,\psi}f(x)\right)
$$
\n
$$
f_{\psi}^{[N]}(x) = f_{\psi}^{(N)}f(x) := \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}f(x),
$$
\n(3)\nb].\n\n
$$
b.
$$
\n(8)\nb].\n\n
$$
b.
$$
\n(9)\ncos (6) and (7) are continuous on [a,b].\n\n
$$
a \in [a,b], f \in C([a,b])
$$
\n
$$
\left(\frac{u}{\psi'(x)}\frac{d}{dx}\right)^{N} \int_{a}^{x} \psi'(t)(\psi(x)-\psi(t))^{N-\mu-1} = E_{\rho,N-\mu}^{-y} \left[\omega(\psi(x)-\psi(t))^{\rho}\right]f(t)dt,
$$
\n(9)\n\n
$$
\left(\frac{u}{\psi'(x)}\frac{d}{dx}\right)^{N} \int_{a}^{x} \psi'(t)(\psi(x)-\psi(t))^{N-\mu-1} = E_{\rho,N-\mu}^{-y} \left[\omega(\psi(x)-\psi(t))^{\rho}\right]f(t)dt,
$$
\n(9)\n\n
$$
f_{\psi}^{[N]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N} \int_{a}^{b} f_{\psi}^{k}f(x)dx dx + \int_{a}^{b} f_{\psi}^{k}f(x)dx dx + \int_{a}^{b} f_{\psi}^{k}f(x)dx dx + \int_{a}^{c} f_{\psi}^{k}f(x)dx dx
$$
\n(10)

and

$$
f_{w}^{[N]}(x) = f_{w}^{(N)} f(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^{N} f(x),
$$
\n(8)  
\n*b*].  
\n
$$
b \in \mathbb{R}
$$
\n(9)  
\n1005 (6) and (7) are continuous on [*a*, *b*].  
\nWe define the  $\psi$ -Prabhakar-Riemann Liouville left and right fractional derivatives of order  $\mu$  (14) as  
\n
$$
\mathbb{E}[a, b], f \in C([a, b]):
$$
\n
$$
\left(\frac{u}{\psi(x)} \frac{d}{dx}\right)^{N} \int_{a}^{x} \psi'(t) (\psi(x) - \psi(t))^{N-\mu-1} E_{\rho, N-\mu}^{-\nu} [\omega(\psi(x) - \psi(t))^{\rho}] f(t) dt,
$$
\n(9)  
\n
$$
\left(\frac{u}{\psi(x)} \frac{d}{dx} \frac{d}{dx}\right)^{N} \int_{x}^{x} \psi'(t) (\psi(t) - \psi(x))^{N-\mu-1} E_{\rho, N-\mu}^{-\nu} [\omega(\psi(t) - \psi(x))^{\rho}] f(t) dt.
$$
\n(10)  
\n110  
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\n11

That is we have

$$
-\frac{1}{\psi'(x)}\frac{d}{dx}\int_{x}^{\psi} \psi'(t)(\psi(t)-\psi(x))^{N-\mu-1} E_{\rho,N-\mu}^{-\nu}[\omega(\psi(t)-\psi(x))^{\rho}]f(t)dt.
$$
\n(10)  
\n
$$
\left(\binom{kL}{\rho,\mu,\omega,a+}f(x)\right) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\left(e_{\rho,N-\mu,\omega,a+}^{-\gamma,\nu}f(x)\right),
$$
\n(11)  
\n
$$
\left(\binom{kL}{\rho,\mu,\omega,b-}f(x)\right) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\left(e_{\rho,N-\mu,\omega,b-}^{-\gamma,\nu}f(x)\right),
$$
\n(12)  
\n
$$
r
$$
-Prabhakar left and right fractional derivatives of order  $\mu$  and type  $0 \le \beta \le 1$  (14),  
\n
$$
\int_{\omega,\omega,a+}^{\psi} f(x) = e_{\rho,\beta(N-\mu),\omega,a+}^{-\gamma,\beta} \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N} e_{\rho,(1-\beta)(N-\mu),\omega,a+}^{-\gamma(1-\beta)\psi}f(x),
$$
\n(13)  
\n(14)  
\n(16)

and

$$
\left(^{RL}D_{\rho,\mu,\omega,b-}^{\gamma;\psi}f\right)(x) = \left(-\frac{1}{\psi^{'}(x)}\frac{d}{dx}\right)^{N}\left(e_{\rho,N-\mu,\omega,b-}^{-\gamma;\psi}f\right)(x),\tag{12}
$$

 $\forall x \in [a,b].$ 

We define also the  $\psi$  -Hilfer-Prabhakar left and right fractional derivatives of order  $\mu$  and type  $0 \le \beta \le 1$  ([4]), as follows

$$
(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right) \int_x \psi(t)\psi(t) - \psi(x)\psi(t) \qquad E_{\rho, N-\mu}[\omega(\psi(t) - \psi(x)) \psi(t)\mu(t)] \qquad (10)
$$
\n
$$
\left(\frac{\kappa L}{\psi(x)}\frac{\partial}{\partial x}\right) \int_x \psi(t)\psi(t) - \psi(x)\psi(t) \qquad (11)
$$
\n
$$
\left(\frac{\kappa L}{\psi(x)}\frac{\partial}{\partial x}\right) \int_x \psi(t)\psi(t) \frac{d}{dx}\psi(t)\psi(t) \qquad (12)
$$
\n
$$
\left(\frac{\kappa L}{\psi(x)}\frac{\partial}{\partial x}\right) \int_x \psi(t)\psi(t)\psi(t)\psi(t)\psi(t)\psi(t)\psi(t)\n\qquad (13)
$$
\n
$$
\psi + \text{Hilfer-Prabhakar left and right fractional derivatives of order }\mu \text{ and type } 0 \le \beta \le 1 \text{ (I4]},
$$
\n
$$
\left(\frac{d}{\psi(x)}\frac{\partial}{\partial x}\right) \int_x \psi(t)\psi(t)\psi(t)\psi(t)\psi(t)\n\qquad (14)
$$
\n
$$
\left(\frac{d}{\psi(x)}\frac{\partial}{\partial x}\right) \int_x \psi(t)\psi(t)\psi(t)\psi(t)\n\qquad (15)
$$
\n
$$
\left(\frac{d}{\psi(x)}\frac{\partial}{\partial x}\right) \int_x \psi(t)\psi(t)\psi(t)\psi(t)\n\qquad (16)
$$

and

$$
\left(\begin{array}{cc} \n\ell^{1} & D_{\rho,\mu,\omega,a+}^{y;\psi} f(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^{N} \left(e_{\rho,N-\mu,\omega,a+}^{-y;\psi} f(x)\right), & (11)
$$
\n
$$
\left(\begin{array}{cc} \n\ell^{1} & D_{\rho,\mu,\omega,b-}^{y;\psi} f(x) = \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^{N} \left(e_{\rho,N-\mu,\omega,b-}^{-y;\psi} f(x)\right), & (12)
$$
\n
$$
\psi \text{ -Hilfer-Prabhakar left and right fractional derivatives of order } \mu \text{ and type } 0 \leq \beta \leq 1 \text{ (I4)}, \quad \text{(II)}
$$
\n
$$
\left(\begin{array}{cc} \n\ell^{1} & D_{\rho,\mu,\omega,a+}^{y;\psi} f(x) = e_{\rho,\beta}^{-y;\psi} f(x) \frac{d}{dx} \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^{N} e_{\rho,(1-\beta)(N-\mu),\omega,a+}^{-y(1-\beta)\psi} f(x), & (13)
$$
\n
$$
\left(\begin{array}{cc} \n\ell^{1} & D_{\rho,\mu,\omega,b-}^{y;\psi} f(x) = e_{\rho,\beta(N-\mu),\omega,b-}^{-y\beta;\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx}\right)^{N} e_{\rho,(1-\beta)(N-\mu),\omega,b-}^{-y(1-\beta)\psi} f(x), & (14)
$$
\n\end{array}\right)

 $\forall x \in [a,b].$ 

When  $\beta = 0$ , we get the Riemann-Liouville version, and when  $\beta = 1$ , we get the Caputo version.

We call 
$$
\xi = \mu + \beta(N - \mu)
$$
, we have that  $N - 1 \le \mu \le \mu + \beta(N - \mu) \le \mu + N - \mu = N$ , hence  $\lceil \xi \rceil = N$ .

We can easily write that

**Fractional Inequalities**

\n**George A. Anastassiou**

\nann-Liouville version, and when 
$$
\beta = 1
$$
, we get the Caputo version.

\nhave that  $N - 1 < \mu \leq \mu + \beta \left( N - \mu \right) \leq \mu + N - \mu = N$ , hence  $\left[ \xi \right] = N$ .

\n
$$
\left( {}^{H}D_{\rho,\mu,\omega,a+}^{\gamma,\beta;\psi} f(x) \right) = e_{\rho,\xi-\mu,\omega,a+}^{-\gamma\beta;\psi} D_{\rho,\xi,\omega,a+}^{\gamma(1-\beta);\psi} f(x),
$$
\n(15)

\n
$$
\left( {}^{H}D_{\rho,\mu,\omega,b-}^{\gamma,\beta;\psi} f(x) \right) = e_{\rho,\xi-\mu,\omega,b-}^{-\gamma\beta;\psi} D_{\rho,\xi,\omega,b-}^{\gamma(1-\beta);\psi} f(x),
$$
\n(16)

\nIt variety of fractional inequalities of Hardy type involving convexity and engaging.

and

$$
\left({}^{H}\mathbf{D}_{\rho,\mu,\omega,b}^{\gamma,\beta;\psi}-f\right)(x)=e_{\rho,\xi-\mu,\omega,b}^{-\gamma\beta;\psi}D_{\rho,\xi,\omega,b}^{\gamma(1-\beta);\psi}f(x),\tag{16}
$$

 $\forall x \in [a,b].$ 

In this work we develop a great variety of fractional inequalities of Hardy type invovling convexity and engaging the above exposed:  $\psi$  -Prabhakar fractional left and right fractional integrals, the  $\psi$  -Prabhakar-Caputo left and right fractional derivatives, the  $\psi$ -Riemann-Liouville left and right fractional derivatives, and the  $\psi$ -Hilfer-Prabhakar left and right fractional derivatives. The radial multivariate case of all of the above over a spherical shell is studied in full detail. We involve ratios of functions and of integral operators and we produce among others vectorial splitting rational  $L_p$  inequalities, as well as separating rational  $L_p$  inequalities.

### 2. Prerequisites

Let  $(\Omega_1, \Sigma_1, \mu_1)$  and  $(\Omega_2, \Sigma_2, \mu_2)$  be measure spaces with positive  $\sigma$ -finite measures, and let  $k$  :  $\Omega_{\rm l}$   $\times$   $\Omega_{\rm 2}$   $\rightarrow$   $\sf R$  be nonnegative measurable functions,  $\,k(x,\cdot)\,$  measurable on  $\,\Omega_{\rm 2}$  , and

$$
K(x) = \int_{\Omega_2} k(x, y) d\mu_2(y), \text{ for any } x \in \Omega_1.
$$
 (17)

We suppose that  $K(x)\!>\!0$  a.e. on  $\Omega_{_1}$  and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions  $g_i:\Omega_1\to\mathsf{R}$ ,  $i=1,...,n,$  with the representation

$$
g_i(x) = \int_{\Omega_2} k(x, y) f_i(y) d\mu_2(y), \qquad (18)
$$

where  $f_i : \Omega_2 \to \mathsf{R}$  are measurable functions,  $i = 1,...,n$ .

Denote by 
$$
\vec{x} = x := (x_1, ..., x_n) \in \mathbb{R}^n
$$
,  $\vec{g} := (g_1, ..., g_n)$  and  $\vec{f} := (f_1, ..., f_n)$ .

We consider here  $\Phi$  :  $R_+^n$   $\to$   $R$  a convex function, which is increasing per coordinate, i.e. if  $x_i \le y_i$ ,  $i=1,...,n$ , then

$$
\Phi(x_1,...,x_n) \le \Phi(y_1,...,y_n).
$$

In [3], p. 588, we proved that

**Theorem 1** Let  $u$  be a weight function on  $\Omega_1$ , and  $k, K, g_i, f_i, i = 1,...,n \in \mathbb{N}$ , and  $\Phi$  defined as above. We suppose that K(x)>0 a.e. on  $s_{i_1}$  and by a weight function (shortly, a weight), we heart a not<br>measurable function on the actual set. Let the measurable functions  $g_i : \Omega_1 \rightarrow \mathbb{R}$ ,  $i = 1,...,n$ ,<br>representation<br> $g_i : \Omega_$ al set. Let the measurable functions  $g_i : \Omega_1 \to \mathbb{R}$ ,  $i = 1,...,n$ , with t<br>  $g_i(x) = \int_{\Omega_2} k(x, y) f_i(y) d\mu_2(y)$ ,<br>
(inntions,  $i = 1,...,n$ .<br>
",  $\overline{g} := (g_1,...,g_n)$  and  $\overline{f} := (f_1,...,f_n)$ .<br>
convex function, which is increasing per c  $f(x,y) = x \mapsto u(x) \frac{k(x,y)}{k(x)}$  is integrable on  $\Omega_1$  for each fixed  $y \in \Omega_2$ . Define  $v$  on  $\Omega_2$  by

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\n
$$
v(y) := \int_{\Omega_1} u(x) \frac{k(x, y)}{K(x)} d\mu_1(x) < \infty.
$$
\n(19)  
\n
$$
|g_n(x)|_{x=0} \int_{\Omega_1} (x) \frac{k(x, y)}{K(x)} d\mu_1(x) dx \leq v.
$$
\n(20)

Then

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\n
$$
v(y) := \int_{\Omega_1} u(x) \frac{k(x, y)}{K(x)} d\mu_1(x) < \infty.
$$
\n(19)  
\n
$$
\int_{\Omega_1} u(x) \Phi\left(\frac{|g_1(x)|}{K(x)}, \dots, \frac{|g_n(x)|}{K(x)}\right) d\mu_1(x) \le \int_{\Omega_2} v(y) \Phi\left(|f_1(y)|, \dots, |f_n(y)|\right) d\mu_2(y),
$$
\n(20)  
\ntions:  
\n
$$
|f_n|, \text{ are } k(x, y) d\mu_2(y) \text{ -integrable, } \mu_1 \text{ -a.e. in } x \in \Omega_1, \text{ for all } i = 1, \dots, n,
$$
\n
$$
|f_n(y)| \text{ is } \mu_2 \text{ -integrable.}
$$
\n
$$
\text{m now on we may write}
$$
\n
$$
\vec{g}(x) = \int_{\Omega_2} k(x, y) \vec{f}(y) d\mu_2(y), \qquad (21)
$$
\n
$$
(g_1(x), \dots, g_n(x)) = \left(\int_{\Omega_2} k(x, y) f_1(y) d\mu_2(y), \dots, \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y)\right).
$$
\n(22)  
\n
$$
|g(x)| = \left|\int_{\Omega_2} k(x, y) \vec{f}(y) d\mu_2(y), \dots, \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y)\right).
$$
\n(23)

under the assumptions:

(i) 
$$
f_i
$$
,  $\Phi(f_1|,...,|f_n|)$ , are  $k(x, y)d\mu_2(y)$  -integrable,  $\mu_1$  -a.e. in  $x \in \Omega_1$ , for all  $i = 1,...,n$ ,

(ii)  $\nu(y) \Phi( \vert f_{1}(y) \vert ,..., \vert f_{n}(y) \vert )$  is  $\mu_{2}$  -integrable.

Notation 2 From now on we may write

$$
\vec{g}(x) = \int_{\Omega_2} k(x, y) \vec{f}(y) d\mu_2(y),\tag{21}
$$

which means

$$
u(x)\Phi\left(\frac{x}{K(x)},...,\frac{x}{K(x)}\right)d\mu_1(x) \leq \int_{\Omega_2} v(y)\Phi\left(y_1(y),...,y_n(y)\right)d\mu_2(y),
$$
  
\nns:  
\n
$$
\text{In, } k(x,y)d\mu_2(y) \text{ -integrable, } \mu_1 \text{ -a.e. in } x \in \Omega_1, \text{ for all } i=1,...,n,
$$
  
\n
$$
\text{In, } k(x,y)d\mu_2(y) \text{ -integrable.}
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y),
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y),
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{f}(y)d\mu_2(y), \text{ and } k(x,y)\overline{f}(y)d\mu_2(y).
$$
  
\n
$$
\text{In, } k(x,y)\overline{
$$

Similarly, we may write

$$
\vec{g}(x) = \left| \int_{\Omega_2} k(x, y) \vec{f}(y) d\mu_2(y) \right|,
$$
\n(23)

and we mean

$$
f_n(f_n(y)) = k(x, y)d\mu_2(y) - \text{integrable}, \mu_1 - \text{a.e. in } x \in \Omega_1, \text{ for all } i = 1,...,n,
$$
  
\n
$$
f_n(y)\text{ is } \mu_2 \text{ -integrable.}
$$
  
\n
$$
\text{Now on we may write}
$$
  
\n
$$
\vec{g}(x) = \int_{\Omega_2} k(x, y) \vec{f}(y) d\mu_2(y),
$$
  
\n
$$
(g_1(x), ..., g_n(x)) = \left( \int_{\Omega_2} k(x, y) f_1(y) d\mu_2(y), ..., \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y) \right).
$$
  
\n
$$
\text{Write}
$$
  
\n
$$
|\vec{g}(x)| = \left| \int_{\Omega_2} k(x, y) \vec{f}(y) d\mu_2(y),
$$
  
\n
$$
(\left| g_1(x), ..., |g_n(x) \right|) = \left( \left| \int_{\Omega_2} k(x, y) f_1(y) d\mu_2(y), ..., \left| \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y) \right| \right) \right).
$$
  
\n
$$
\text{that}
$$
  
\n
$$
|g(x)| = \left( \int_{\Omega_2} k(x, y) f_1(y) d\mu_2(y), ..., \left| \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y) \right| \right).
$$
  
\n
$$
\text{that}
$$
  
\n
$$
|g(x)| = \left( \int_{\Omega_2} k(x, y) \vec{f}(y) d\mu_2(y), ..., \left| \int_{\Omega_2} k(x, y) f_n(y) d\mu_2(y) \right| \right).
$$
  
\n(25)

We also can write that

$$
|\vec{g}(x)| \leq \int_{\Omega_2} k(x, y) |\vec{f}(y)| d\mu_2(y), \tag{25}
$$

and we mean the fact that

$$
|g_i(x)| \leq \int_{\Omega_2} k(x, y) |f_i(y)| d\mu_2(y), \tag{26}
$$

for all  $i = 1,...,n$ , etc.

**Notation 3** Next let  $(\Omega_1, \Sigma_1, \mu_1)$  and  $(\Omega_2, \Sigma_2, \mu_2)$  be measure spaces with positive  $\sigma$  -finite measures, and let  $k_j$  :  $\Omega_1\times\Omega_2\to$   $\sf R\,$  be a nonnegative measurable function,  $k_j(x,\cdot)$  measurable on  $\Omega_2\,$  and

$$
K_j(x) = \int_{\Omega_2} k_j(x, y) d\mu_2(y), \quad x \in \Omega_1, j = 1, \dots, m. \tag{27}
$$

We suppose that  $K_j(x)\ge 0$  a.e. on  $\Omega_1$  . Let the measurable functions  $|g_{ji}:\Omega_1\to\mathsf{R}|$  with the representation

$$
g_{ji}(x) = \int_{\Omega_2} k_j(x, y) f_{ji}(y) d\mu_2(y),
$$
\n(28)

where  $f_{ji}$ : $\Omega_2 \rightarrow \mathsf{R}$  are measurable functions,  $i = 1,...,n$  and  $j = 1,...,m$ .

Denote the function vectors  $\vec{g}_j := (g_{j1}, g_{j2},..., g_{jn})$  and  $\vec{f}_j := (f_{j1},..., f_{jn})$   $j = 1,..., m$ .  $\overline{a}$  $j = 1,...,m$ .

We say  $f_j$  $\overline{a}$ is integrable with respect to measure  $\mu$  , iff all  $f_{ji}$  are integrable with respect to  $\mu$ .

We also consider here  $\Phi_j: \mathsf{R}_+^n {\,\rightarrow\,} \mathsf{R}_+$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}_+$ ,  $\;j$  =  $1,...,m,$  convex functions that are increasing per coordinate. Again  $u$  is a weight function on  $\, \Omega_{\rm 1}^{}$  .

### We make

**Remark 4** Following Notation 3, let  $F_j : \Omega_2 \to \mathbb{R} \cup {\pm \infty}$  be measurable functions,  $j = 1,...,m$ , with  $0\leq F_j(y)\leq\infty$  on  $\Omega_2$  . In (27) we replace  $k_j(x,y)$  by  $k_j(x,y)F_j(y)$ ,  $j=1,...,m,$  and we have the modified  $K_j(x)$ as Notation 3, let  $F_j : \Omega_2 \to \mathbb{R} \cup \{\pm \infty\}$  be measurable functions,  $j = 1,...,m$ , with<br>
(27) we replace  $k_j(x, y)$  by  $k_j(x, y)F_j(y)$ ,  $j = 1,...,m$ , and we have the modified  $K_j(x)$ <br>  $L_j(x) := \int_{\Omega_2} k_j(x, y)F_j(y) d\mu_2(y)$ ,  $x \in \Omega_1$ .<br>
(29  $\mathbf{R} \cup \{\pm \infty\}$  be measurable functions,  $j = 1,...,m$ , with<br>  $\mathbf{r}_j(x, y)F_j(y)$ ,  $j = 1,...,m$ , and we have the modified  $K_j(x)$ <br>  $F_j(y)d\mu_2(y), x \in \Omega_1$ . (29)<br>
there  $\vec{f}_j = (f_{j1},..., f_{j_n})$ ;  $\vec{p} = \left(\frac{f_{j1}}{F_j}, ..., \frac{f_{j_n}}{F_j}\right)$ <br>  $=$ 

$$
L_j(x) := \int_{\Omega_2} k_j(x, y) F_j(y) d\mu_2(y), x \in \Omega_1.
$$
 (29)

We assume  $L_j(x)\!>\!0$  a.e. on  $\Omega_1$  .

As new 
$$
\vec{f}_j
$$
 we consider now  $\vec{y} := \frac{\vec{f}_j}{F_j}$ ,  $j = 1,...,m$ , where  $\vec{f}_j = (f_{j1},...,f_{jn})$ ;  $\vec{y} = (\frac{f_{j1}}{F_j},...,\frac{f_{jn}}{F_j})$ .

Notice that

$$
g_{ji}(x) = \int_{\Omega_2} k_j(x, y) f_{ji}(y) d\mu_2(y) = \int_{\Omega_2} (k_j(x, y) F_j(y)) \left( \frac{f_{ji}(y)}{F_j(y)} \right) d\mu_2(y), \tag{30}
$$

 $x \in \Omega_{1, all}$  j = 1,...,*m*; i = 1,...,*n*.

So we can write

$$
\vec{g}_j(x) = \int_{\Omega_2} \left( k_j(x, y) F_j(y) \right) \vec{g}(y) d\mu_2(y), \ \ j = 1, ..., m. \tag{31}
$$

We mention

**Theorem 5** ([3], p. 481) Here we follow Remark 4. Let  $\rho \in \{1,...,m\}$  be fixed. Assume that the function

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$$
x \mapsto \left(\frac{u(x) \left(\prod_{j=1}^{m} F_j(y)\right) \left(\prod_{j=1}^{m} k_j(x, y)\right)}{\prod_{j=1}^{m} L_j(x)}\right)
$$
\nDefine  $U_m$  on  $\Omega_2$  by

is integrable on  $\Omega_{\text{\tiny I}}$  , for each  $\,y\!\in\!\Omega_{\text{\tiny 2}}$  . Define  $\,U_{_{m}}\,$  on  $\,\Omega_{\text{\tiny 2}}\,$  by

 <sup>&</sup>lt; . , := <sup>1</sup> =1 =1 =1 1 d x L x u x k x y U y F y j m j j m j j m j <sup>m</sup> (32) , <sup>2</sup>

Then

$$
\int_{\Omega_1} u(x) \prod_{j=1}^m \Phi_j \left( \left| \frac{\overrightarrow{g_j}(x)}{L_j(x)} \right| \right) d\mu_1(x) \leq \left( \prod_{\substack{j=1 \ j \neq \rho}}^m \int_{\Omega_2} \Phi_j \left( \left| \frac{\overrightarrow{f_j}(y)}{F_j(y)} \right| \right) d\mu_2(y) \right) \cdot \left( \int_{\Omega_2} \Phi_\rho \left( \left| \frac{\overrightarrow{f_\rho}(y)}{F_\rho(y)} \right| \right) U_m(y) d\mu_2(y) \right), \tag{33}
$$

under the assumptions:

der the assumptions:  
\n(i) 
$$
\overrightarrow{f}
$$
,  $\Phi_j\left(\frac{|\overrightarrow{f}|}{F_j}\right)$  are both  $k_j(x, y)F_j(y)d\mu_2(y)$  -integrable,  $\mu_1$  -a.e. in  $x \in \Omega_1$ ,  $j = 1,...,m$ ,  
\n(ii)  $U_m\Phi_p\left(\frac{\overrightarrow{f}}{F_p}\right)$ ;  $\Phi_1\left(\frac{|\overrightarrow{f}|}{F_1}\right)$ ,  $\Phi_2\left(\frac{|\overrightarrow{f}|}{F_2}\right)$ ,...,  $\Phi_p\left(\frac{|\overrightarrow{f}|}{F_p}\right)$ ,...,  $\Phi_p\left(\frac{|\overrightarrow{f}|}{F_p}\right)$ , are  $\mu_2$ -integrable, where  $\Phi_p\left(\frac{|\overrightarrow{f}|}{F_p}\right)$  is absent.  
\nWe also mention  
\n**Theorem 6** ((3], p. 519) Here all as in Notation 3 and Remark 4. Assume that the functions  $(j = 1, 2,..., m \in \mathbb{N}$ )  
\n $x \mapsto \left(\frac{u(x)k_j(x, y)F_j(y)}{K_j(x)}\right)$   
\n $\vdots$  integrable on  $\Omega_1$ , for each fixed  $y \in \Omega_2$ . Define  $W_j$  on  $\Omega_2$  by  
\n
$$
W_j(y):=\left(\int_{\Omega_1}\frac{u(x)k_j(x, y)}{K_j(x)}d\mu_1(x)\right)F_j(y) < \infty,
$$
\n(34)  
\n $\Omega_2$ .

We also mention

**Theorem 6** ([3], p. 519) Here all as in Notation 3 and Remark 4. Assume that the functions ( $j = 1,2,...,m \in \mathbb{N}$ )

$$
x \mapsto \left(\frac{u(x)k_j(x, y)F_j(y)}{K_j(x)}\right)
$$

are integrable on  $\Omega_{\text{\tiny{1}}}$  , for each fixed  $\ y\,{\in}\,\Omega_{\text{\tiny{2}}}$  . Define  $W_j$  on  $\Omega_{\text{\tiny{2}}}$  by

$$
W_j(y) := \left(\int_{\Omega_1} \frac{u(x)k_j(x, y)}{K_j(x)} d\mu_1(x)\right) F_j(y) < \infty,
$$
\n(34)

on  $\Omega_2$ .

Let  $p_i > 1: \sum_{i=1}^{m} \frac{1}{i} = 1$ .  $=$   $\mu_j$ m j  $p_j > 1$  :  $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$  . Let the functions  $\Phi_j : \mathsf{R}_+^n \to \mathsf{R}_+$  $j_{ij}$  :  $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}_+$ ,  $\;j$  =  $1,...,m,$  be convex and increasing per coordinate.

Then

1. Let the functions 
$$
\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+
$$
,  $j = 1, \ldots, m$ , be convex and increasing per coordinate.

\n
$$
\frac{1}{p_j} = 1.
$$
 Let the functions  $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$ ,  $j = 1, \ldots, m$ , be convex and increasing per coordinate.\n
$$
\int_{\Omega_1} u(x) \prod_{j=1}^m \Phi_j \left( \left| \frac{\overrightarrow{g}(x)}{L_j(x)} \right| \right) d\mu_1(x) \leq \prod_{j=1}^m \left( \int_{\Omega_2} W_j(y) \Phi_j \left( \left| \frac{\overrightarrow{f}(y)}{F_j(y)} \right| \right)^{p_j} d\mu_2(y) \right)^{\frac{1}{p_j}},
$$
\n(35)

\n2. Show that,  $\sum_{j=1}^n \Phi_j \left( \frac{\overrightarrow{f}(y)}{F_j(y)} \right)^{p_j} d\mu_2(y)$ , where  $\sum_{j=1}^n \Phi_j$  is the same as the function  $\Phi_j$  and  $\Phi_j$  is the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and  $\Phi_j$  are the same as the function  $\Phi_j$  and 

under the assumptions:

(i) 
$$
\overrightarrow{\hat{f}}_j
$$
,  $\Phi_j \left( \frac{|\overrightarrow{f}|}{F_j} \right)^{p_j}$  are both  $k_j(x, y)F_j(y) d\mu_2(y)$  -integrable,  $\mu_1$  -a.e. in  $x \in \Omega_1$ ,  $j = 1,...,m$ ,  
\n(ii)  $W_j \Phi_j \left( \frac{|\overrightarrow{f}|}{F_j} \right)^{p_j}$  is  $\mu_2$  -integrable,  $j = 1,...,m$ .

We make

**Remark 7** Let  $(\Omega_1, \Sigma_1, \mu_1)$  and  $(\Omega_2, \Sigma_2, \mu_2)$  be measure spaces with positive  $\sigma$ -finite measures, and let  $k$  :  $\Omega_{\rm l}$   $\times$   $\Omega_{\rm 2}$   $\rightarrow$   $\sf R$  be nonnegative measurable functions,  $\,k(x,\cdot)\,$  measurable on  $\,\Omega_{\rm 2}$  , and

$$
K(x) = \int_{\Omega_2} k(x, y) d\mu_2(y), \text{ for any } x \in \Omega_1.
$$

We assume  $\,K\!(x)\!>\!0\,$  a.e. on  $\,\Omega_{_1}\,$  and the weight functions are nonnegative functions on the related set. We consider measurable functions  $\,g_{\,i}:\!\Omega_{1} \!\rightarrow\! {\sf R}$  , with the representation

$$
g_i(x) = \int_{\Omega_2} k(x, y) f_i(y) d\mu_2(y),
$$

where  $f_i:\Omega_2\to\mathsf{R}$  are measurable functions,  $i=1,...,n$  . Here  $u$  stands for a weight function on  $\Omega_1.$  So we follow Notation 3 for  $j$  =  $m$  =  $1$  . We write here  $\,\vec{g}:=(g_1,...,g_n),\,\vec{f}:=(f_1,...,f_n).$  $\overline{a}$ .

We set

$$
Q_2, \Sigma_2, \mu_2
$$
 be measure spaces with positive  $\sigma$ -finite measures, and let  
variable functions,  $k(x, \cdot)$  measurable on  $\Omega_2$ , and  
 $K(x) = \int_{\Omega_2} k(x, y) d\mu_2(y)$ , for  
any  $\in \Omega_1$ .  
and the weight functions are nonnegative functions on the related set. We  
 $\rightarrow R$ , with the representation  
 $g_i(x) = \int_{\Omega_2} k(x, y) f_i(y) d\mu_2(y)$ ,  
functions,  $i = 1,...,n$ . Here *u* stands for a weight function on  $\Omega_1$ . So we  
ite here  $\vec{g} := (g_1,...,g_n)$ ,  $\vec{f} := (f_1,...,f_n)$ .  

$$
\left\| \vec{f}(y) \right\|_{\infty} := \max \{ f_1(y), ..., |f_n(y) \},
$$
and  

$$
\left\| \vec{f}(y) \right\|_{q} := \left( \sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, q \ge 1.
$$
 (36)

We assume that

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$$
0 < \left\| \vec{f}(y) \right\|_q < \infty, \text{ a.e.} \text{on}(a, b), \tag{37}
$$

 $1 \leq q \leq \infty$  fixed.

Let

Journal of Advances in Applied & Computational Mathematics, 8, 2021\n
$$
0 < \left\| \vec{f}(y) \right\|_q < \infty, \text{ a.e.} \text{on}(a, b), \tag{37}
$$
\n
$$
L_q(x) := \int_{\Omega_2} k(x, y) \left\| \vec{f}(y) \right\|_q d\mu_2(y), \, x \in \Omega_1, \tag{38}
$$

 $1 \leq q \leq \infty$  fixed.

We assume  $L_q(x) > 0$  a.e. on  $\Omega_1$ .

We furher assume that the function

Journal of Advances in Applied & Computational Mathematics, 8, 2021\n
$$
0 < \left\| \vec{f}(y) \right\|_q < \infty, \text{ a.e.} \text{on}(a, b), \tag{37}
$$
\n
$$
L_q(x) := \int_{\Omega_2} k(x, y) \left\| \vec{f}(y) \right\|_q d\mu_2(y), x \in \Omega_1, \tag{38}
$$
\n
$$
x \mapsto \left( \frac{u(x)k(x, y) \left\| \vec{f}(y) \right\|_q}{L_q(x)} \right) \tag{39}
$$
\nfixed  $y \in \Omega_2$ .

\n(y): = 
$$
\left( \int_{\Omega_1} \frac{u(x)k(x, y)}{L_q(x)} d\mu_1(x) \right) \left\| \vec{f}(y) \right\|_q < \infty, \tag{40}
$$

is integrable on  $\, \Omega_{\!1}^{}$ , for almost each fixed  $\, y \, \epsilon \Omega_{2}^{} . \,$ 

Define  $W_q$  on  $\Omega_2$  by

$$
W_q(y) := \left( \int_{\Omega_1} \frac{u(x)k(x, y)}{L_q(x)} d\mu_1(x) \right) \left\| \vec{f}(y) \right\|_q < \infty,
$$
\n(40)

a.e. on  $\Omega_2$ .

Let

$$
x \mapsto \left(\frac{u(x)k(x,y) \| f(y) \|_{q}}{L_{q}(x)}\right)
$$
\n(39)

\nLet  $\Omega_{1}$ , for almost each fixed  $y \in \Omega_{2}$ .

\n
$$
W_{q} \text{ on } \Omega_{2} \text{ by}
$$
\n
$$
W_{q}(y) := \left(\int_{\Omega_{1}} \frac{u(x)k(x,y)}{L_{q}(x)} d\mu_{1}(x)\right) \left\|\vec{f}(y)\right\|_{q} < \infty,
$$
\n
$$
\vec{y} := \left(\frac{f_{1}}{\left\|\vec{f}(y)\right\|_{q}}, \frac{f_{2}}{\left\|\vec{f}(y)\right\|_{q}}, \dots, \frac{f_{n}}{\left\|\vec{f}(y)\right\|_{q}}\right),
$$
\n
$$
\vec{y} := \left(\frac{f_{1}}{\left\|\vec{f}(y)\right\|_{q}}, \frac{f_{2}}{\left\|\vec{f}(y)\right\|_{q}}, \dots, \frac{f_{n}}{\left\|\vec{f}(y)\right\|_{q}}\right),
$$
\n
$$
\left(\frac{y}{\left\|\vec{f}(y)\right\|_{q}}, \dots, \frac{y}{\left\|\vec{f}(y)\right\|_{q}}\right),
$$
\n
$$
\left(\frac{y}{\left\|\vec{f}(y)\right\|_{q}}, \dots, \frac{y}{\left\|\vec{f}(y)\right\|_{q}}\right),
$$
\n(41)

\nTherefore, the following equation for  $\Omega_{1}$  and  $\Omega_{2}$  is a convex and increasing per coordinate function.

i.e. 
$$
\vec{y} = \frac{\vec{f}}{\left\| \vec{f}(y) \right\|_q}
$$
.

Here  $\Phi$  :  $R_{+}^n$   $\rightarrow$   $R$  is a convex and increasing per coordinate function.

We mention

Theorem 8 ([3], p. 536) Let all here as in Remark 7. Then

smallized Fractional Inequalities

\n
$$
\int_{\Omega_1} u(x) \Phi\left(\left|\frac{\vec{g}(x)}{L_q(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} W_q(y) \Phi\left(\left|\frac{\vec{f}(y)}{\vec{f}(y)}\right|_q\right) d\mu_2(y),\tag{42}
$$

under the assumptions:

Corial Prabhakar Hardy Type Generalized Fractional Inequalities

\nGeorge A. Anastassiou

\n
$$
\int_{\Omega_1} u(x) \Phi\left(\left|\frac{\vec{g}(x)}{L_q(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} W_q(y) \Phi\left(\left|\frac{\vec{f}(y)}{|\vec{f}(y)|_q}\right| d\mu_2(y),
$$
\nHere the assumptions:

\n(i) 
$$
\frac{\vec{f}(y)}{\left\|\vec{f}(y)\right\|_q}
$$

\n, 
$$
\Phi\left(\left|\frac{\vec{f}(y)}{\left\|\vec{f}(y)\right\|_q}\right)
$$

\nare both  $k(x, y) \left\|\vec{f}(y)\right\|_q d\mu_2(y)$ -integrable,  $\mu_1$ -a.e. in  $x \in \Omega_1$ ,

\n(ii) 
$$
W_q(y) \Phi\left(\left|\frac{\vec{f}(y)}{\left\|\vec{f}(y)\right\|_q}\right)
$$
 is  $\mu_2$ -integrable.

\nTheorem 8 comes directly from Theorem 1.

\nWe will also use:

Theorem 8 comes directly from Theorem 1.

We will also use:

Let  $(\Omega_1,\Sigma_1,\mu_1)$ ,  $(\Omega_2,\Sigma_2,\mu_2)$  measure spaces with positive  $\sigma$ -finite measures, and  $k_i$  :  $\Omega_1\times\Omega_2\to$  R are nonnegative measurable functions, with  $k_i(x,\cdot)$  measurable on  $\Omega_2$  , and measurable functions  $\,g_{\,ji}:\!\Omega_1\to\!{\sf R}\,$  : ssitive  $\sigma$ -finite measures, and  $k_i : \Omega_1 \times \Omega_2 \to \mathsf{R}$  are<br>
on  $\Omega_2$ , and measurable functions  $g_{ji} : \Omega_1 \to \mathsf{R}$ :<br>  $f_{ji}(y)d\mu_2(y)$ ,<br>
2;  $i = 1,...,m$ .<br>
..,*m*. Assume that the functions ( $i = 1,...,m \in \mathbb{N}$ )<br>  $\frac{y)f_{2i}(y)}{($ easure spaces with positive  $\sigma$ -finite measures, and  $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$  are<br>
ith  $k_i(x, \cdot)$  measurable on  $\Omega_2$ , and measurable functions  $g_{ji} : \Omega_1 \to \mathbb{R}$ :<br>  $g_{ji}(x) = \int_{\Omega_2} k_i(x, y) f_{ji}(y) d\mu_2(y)$ ,<br>
functions, for a sitive  $\sigma$ -finite measures, and  $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$  are<br>
on  $\Omega_2$ , and measurable functions  $g_{ji} : \Omega_1 \to \mathbb{R}$ :<br>  $f_{ji}(y) d\mu_2(y)$ ,<br>  $2; i = 1,...,m$ .<br>
, m. Assume that the functions  $(i = 1,...,m \in \mathbb{N})$ <br>  $\frac{y}{j_2(y)}$ <br>
, a.e

$$
g_{ji}(x) = \int_{\Omega_2} k_i(x, y) f_{ji}(y) d\mu_2(y),
$$

where  $f_{ji}$  :  $\Omega_2 \rightarrow \mathsf{R}$  are measurable functions, for all  $j = 1, 2; i = 1, ..., m$ .

**Theorem 9** ([3], p. 552) Here  $0 < f_{2i}(y) < \infty$  , a.e.,  $i = 1,...,m$ . Assume that the functions ( $i = 1,...,m \in \mathbb{N}$ )

$$
x \mapsto \left(\frac{u(x)k_{i}(x, y)f_{2i}(y)}{g_{2i}(x)}\right)
$$

are integrable on  $\, \Omega_{_1}$ , for each fixed  $\, y \in \Omega_{_2} ;$  with  $\, g_{_{2i}}(x) \! > \! 0$  , a.e. on  $\, \Omega_{_1} . \,$ 

Define  $\psi_{_i}$  on  $\Omega_{_2}$  by

$$
\psi_i(y) := f_{2i}(y) \int_{\Omega_1} u(x) \frac{k_i(x, y)}{g_{2i}(x)} d\mu_1(x) < \infty,
$$
\n(43)

a.e. on  $\Omega_2$ .

Let 
$$
p_i > 1
$$
:  $\sum_{i=1}^{m} \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex and increasing. Then

Journal of Advances in Applied & Computational Mathematics, 8, 2021\n
$$
\int_{\Omega_1} u(x) \prod_{i=1}^m \Phi_i \left( \left| \frac{g_{1i}(x)}{g_{2i}(x)} \right| \right) d\mu_i(x) \leq \prod_{i=1}^m \left( \int_{\Omega_2} \psi_i(y) \Phi_i \left( \left| \frac{f_{1i}(y)}{f_{2i}(y)} \right| \right)^{p_i} d\mu_2(y) \right)^{\frac{1}{p_i}},
$$
\n(44)\n
$$
f_{1i}(y) \Big| \Big|^{p_i}
$$
\n(45)

under the assumptions:

(i) , 2 1 f y f y i <sup>i</sup> i p i i i f y f y 2 1 are both k x yf yd y <sup>i</sup> <sup>2</sup><sup>i</sup> <sup>2</sup> , -integrable, <sup>1</sup> -a.e. in , <sup>1</sup> x (ii) i p i i i i f y f y y 2 <sup>1</sup> is <sup>2</sup> -integrable, i =1,...,m. := := max ,

## 3.Main Results

We make

**Remark 10** Here  $\rho_j, \mu_j, \gamma_j, \omega_j > 0;$   $f_{ji} \in C([a,b])$  and  $\psi \in C^1([a,b])$  which is increasing;  $j=1,...,m$  and  $i = 1, ..., n$ . Set

$$
L_{\infty} \varphi_{j+}(y) := \left\| \overrightarrow{e_{\rho_j, \mu_j, \omega_j, a+}^{j; \psi}} f_j(y) \right\|_{\infty} := \max_{\substack{j=1, \dots, m \\ i=1, \dots, n}} \left\{ \left| e_{\rho_j, \mu_j, \omega_j, a+}^{j; \psi} f_j(y) \right| \right\},\tag{45}
$$

and

$$
\rho_j, \mu_j, \gamma_j, \omega_j > 0; f_{ji} \in C([a, b]) \text{ and } \psi \in C^1([a, b]) \text{ which is increasing; } j = 1, \dots, m \text{ and}
$$
\n
$$
\phi_{j+}(y) := \left\| \overline{e_{\rho_j, \mu_j, \omega_j, a+}^{y, \psi}} f_j(y) \right\|_{\infty} := \max_{i=1, \dots, n} \left\{ \left| e_{\rho_j, \mu_j, \omega_j, a+}^{y, \psi} f_j(y) \right\}, \qquad (45)
$$
\n
$$
\phi_{j+}(y) := \left\| \overline{e_{\rho_j, \mu_j, \omega_j, a+}^{y, \psi}} f_j(y) \right\|_{\infty} := \left\| \sum_{i=1, \dots, n}^{n} \left\{ e_{\rho_j, \mu_j, \omega_j, a+}^{y, \psi} f_j(y) \right\}, \qquad (46)
$$
\n
$$
\text{are continuous functions, } j = 1, \dots, m. \text{ We have that}
$$

 $y\in\!a,b]$ , which  $_{q}\varphi_{_{j+}}$  are continuous functions,  $\ j=1,...,m$  . We have that

$$
0 \leq_q \varphi_{j+}(y) < \infty \text{ in } [a, b], \tag{47}
$$

 $j = 1,...,m$ ; where  $1 \le q \le \infty$  is fixed.

Here it is

$$
\mathcal{L}(\mathcal{P}_{\rho}) = \left\| \mathcal{C}_{\rho_{j},\mu_{j},\omega_{j},a+}^{j,\mathbf{w}} f_{j}(\mathbf{y}) \right\|_{\infty} := \max_{j=1,\dots,n} \left\{ \left| \mathcal{C}_{\rho_{j},\mu_{j},\omega_{j},a+}^{j,\mathbf{w}} f_{j}(\mathbf{y}) \right| \right\},
$$
\n
$$
\mathcal{Q}_{\rho}(\mathbf{y}) := \left\| \mathcal{C}_{\rho_{j},\mu_{j},\omega_{j},a+}^{j,\mathbf{w}} f_{j}(\mathbf{y}) \right\|_{\mathcal{Q}} := \left( \sum_{i=1}^{n} \left| \mathcal{C}_{\rho_{j},\mu_{j},\omega_{j},a+}^{j,\mathbf{w}} f_{j}(\mathbf{y}) \right|^{q} \right) \mathcal{P}_{\mathcal{P}}(\mathbf{y}) = 1,
$$
\n
$$
\text{which } \mathcal{Q}_{\rho_{j}}(\mathbf{y}) = \mathcal{C}_{\mathcal{Q}} \left\| \mathcal{C}_{\rho_{j},\mu_{j},\omega_{j},a+}^{j,\mathbf{w}} f_{j}(\mathbf{y}) \right\|_{\mathcal{Q}} := \left\| \mathcal{C}_{\mathcal{Q}}^{j,\mathbf{w}} f_{j}(\mathbf{y}) \right\|_{\mathcal{Q}} = 1;
$$
\n
$$
\text{which } \mathcal{Q}_{\rho_{j}}(\mathbf{y}) = \mathcal{C}_{\mathcal{Q}} \left\| \mathcal{C}_{\mathcal{Q}}(\mathbf{y}) = 1, \dots, m. \text{ We have that}
$$
\n
$$
0 <_{q} \mathcal{Q}_{j+}(\mathbf{y}) < \infty \text{ in } [a, b].
$$
\n
$$
\text{This } \mathcal{U}_{\mathcal{Q}}(\mathbf{x}) = \mathcal{U}_{\mathcal{Q}}(\mathbf{x}) = \mathcal{U}_{\mathcal{Q}}(\mathbf{y}) \mathcal{U}_{\mathcal{Q}}(\mathbf{y}) = 1.
$$
\n
$$
\mathcal{U}_{\mathcal{Q}}(\mathbf{x}) = \mathcal{U}_{\mathcal{Q}}(\mathbf{x}) = \mathcal{U}_{\mathcal{Q}}(\mathbf{x}) = \mathcal{U}_{\mathcal{Q}}(\mathbf{x}) = \math
$$

 $j = 1,...,m$ , and

$$
L_{jq}^+(x) := \int_a^x \psi'(y) (\psi(x) - \psi(y))^{\mu_j - 1} E_{\rho_j, \mu_j}^{y_j} [\omega_j (\psi(x) - \psi(y))^{\rho_j}]_q \varphi_{j+}(y) dy,
$$
\n
$$
\forall x \in [a, b], 1 \le q \le \infty.
$$
\n(49)

We have that  $\,L_{_{jq}}^*(x)\!>\!0\,$  on  $\big[a,b\big].$ 

Let  $\rho \in \{1,...,m\}$  be fixed. The weight function  $u$  is chosen so that

2 red Fractional Inequalities

\nGeorge A. Anastassiou

\n
$$
[a,b].
$$
\n6eorge A. Anastassiou

\n
$$
[a,b].
$$
\n
$$
U_m^+(y) := \left(\prod_{j=1}^m \varphi_{j+}(y)\right) \int_y^b \frac{u(x) \prod_{j=1}^m k^+_j(x,y)}{\prod_{j=1}^m L^+_{jq}(x)} dx < \infty,
$$
\n(50)

\ntegrable on  $[a,b]$ 

 $\forall\,$   $y$   $\in$   $\left[ a,b\right]$ , and that  $\,{U_{m}^{\text{+}}}$  is integrable on  $\left[ a,b\right]$ 

A direct application of Theorem 5 gives:

**Theorem 11** It is all as in Remark 10. Here  $\Phi_j$  : $\mathsf{R}_+^n$   $\rightarrow$   $\mathsf{R}_+$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$  =  $1,...,m$  , are convex functions increasing per coordinate. Then

ve that 
$$
L_n^*(x) > 0
$$
 on [a, b].  
\n
$$
\in \{1,...,m\} \text{ be fixed. The weight function } u \text{ is chosen so that}
$$
\n
$$
U_n^*(y) := \left(\prod_{j=1}^n \varphi_{j,k}(y)\right) \int_y^{\alpha} \frac{u(x)\prod_{j=1}^m k_j^*(x, y)}{\prod_{j=1}^m k_j^*(x)} dx < \infty,
$$
\n(50)  
\nb], and that  $U_n^*$  is integrable on [a, b].  
\n
$$
\text{the application of Theorem 5 gives:}
$$
\n
$$
\text{cm 11 It is all as in Remark 10. Here } \Phi_j: \mathbb{R}_+^n \to \mathbb{R}_+, j = 1,...,m, \text{ are convex functions increasing per}
$$
\n
$$
\text{cm. Then}
$$
\n
$$
\int_u^b u(x) \prod_{j=1}^m \Phi_j \left(\frac{\overline{e_{j,\mu}^{r,w}}}{\overline{E_{j,q}^{r}(x)}}\right) dx \le \left(\prod_{j=1}^n \int_u^b \Phi_j \left(\frac{\overline{f_j(y)}}{\varphi_{j,k}(y)}\right) dy \right) \left(\int_u^b \Phi_j \left(\frac{\overline{f_j(y)}}{\varphi_{j,k}(y)}\right) U_n^*(y) dy\right).
$$
\n(51)  
\n
$$
\text{like}
$$
\n
$$
\text{the } \mathbb{R} \times \mathbb{R} \text{ are } \rho_j, \mu_j, \gamma_j, \omega_j > 0; f_{ji} \in C([a, b]) \text{ and } \psi \in C^1([a, b]) \text{ which is increasing; } j = 1,...,m \text{ and}
$$
\n
$$
\omega \varphi_j(y) := \left|\overline{e_{j,j,\mu_j,\omega_j,b}^{r,w} f_j(y)}\right|_{\infty} := \max_{i=1,...,n} \left|\overline{e_{j,j,\mu_j,\omega_j,b}^{r,w} f_j(y)}\right|_{\infty} := \left(\sum_{i=1}^n \left|\overline{e_{j,j,\mu_j,\omega_j,b}^{r,w} f_j(y)}\right|_{\infty}^b = \left(\sum_{i=1}^n \left|\overline{e_{j,j,\mu_j,\omega_j,b}^{r,w} f_j(y)}\right|_{\infty}^b = \left(\sum_{i=1}^n \
$$

We make

**Remark 12** Here  $\rho_j, \mu_j, \gamma_j, \omega_j > 0;$   $f_{ji} \in C([a,b])$  and  $\psi \in C^1([a,b])$  which is increasing;  $j=1,...,m$  and  $i = 1,...,n$ . Set

$$
\mathcal{D}_{\varphi}(\mathbf{y}) := \left\| \overrightarrow{e}_{\rho_j, \mu_j, \omega_j, b}^{y_j, \psi} - f_j(\mathbf{y}) \right\|_{\infty} := \max_{\substack{j=1, \dots, m \\ i=1, \dots, n}} \left\{ \left| e_{\rho_j, \mu_j, \omega_j, b}^{y_j, \psi} - f_{ji}(\mathbf{y}) \right| \right\},\tag{52}
$$

and

$$
q\varphi_{j-}(y) := \left\| \overrightarrow{e_{\rho_{j},\mu_{j},\omega_{j},b-}^{y,y}} f_{j}(y) \right\|_{q} := \left( \sum_{i=1}^{n} \left| e_{\rho_{j},\mu_{j},\omega_{j},b-}^{y,y}\ f_{ji}(y) \right|^{q} \right)^{\frac{1}{q}}, q \ge 1; \tag{53}
$$

 $y\in\!a,b]$ , which  $_{q}\varphi_{j-}$  are continuous functions,  $\ j=1,...,m$  . We have also that

$$
0 <_{q} \varphi_{j-}(y) < \infty \text{ in } [a, b], \tag{54}
$$

 $j = 1,...,m$ ; where  $1 \le q \le \infty$  is fixed.

Here it is

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stassiou  
\n
$$
k_j^-(x, y) := k_j(x, y) = \begin{cases}\n\psi'(y)(\psi(y) - \psi(x))^{\mu_j - 1} E_{\rho_j, \mu_j}^{y_j} [\omega_j(\psi(y) - \psi(x)]^{\rho_j}], & x \leq y < b, \\
0, a < y < x,\n\end{cases}
$$
\n(55)  
\n
$$
L_{jq}^-(x) := \int_x^b \psi'(y)(\psi(y) - \psi(x))^{\mu_j - 1} E_{\rho_j, \mu_j}^{y_j} [\omega_j(\psi(y) - \psi(x)]^{\rho_j}]\Big|_q \varphi_{j-}(y) dy,
$$
\n(56)  
\n
$$
k_j^-(x) := \int_x^b \psi'(y)(\psi(y) - \psi(x))^{\mu_j - 1} E_{\rho_j, \mu_j}^{y_j} [\omega_j(\psi(y) - \psi(x)]^{\rho_j}]\Big|_q \varphi_{j-}(y) dy,
$$

 $j = 1,...,m$ , and

$$
L_{jq}^-(x) := \int_x^b \psi'(y) (\psi(y) - \psi(x))^{\mu_j - 1} E_{\rho_j, \mu_j}^{y_j} [\omega_j(\psi(y) - \psi(x))^{\rho_j}]_q \varphi_j(y) dy,
$$
\n(56)

 $\forall x \in [a, b]$ ,  $1 \le q \le \infty$ .

We have that  $\, L^{-}_{jq}(x) \! > \! 0 \,$  on  $\big[a, b\big].$ 

Let  $\rho \in \{1,...,m\}$  be fixed. The weight function  $u$  is chosen so that

$$
\int_{\mathcal{L}} \mathcal{L} \left[ \int_{\mathcal{L}} \mathcal{L} \left( \mathcal{L} \left( \mathcal{L} \right) \mathcal{L} \left( \mathcal{L} \right) \right) \mathcal{L} \left( \mathcal{L} \right) \
$$

 $\forall\,\,y\!\in\!\left[ a,b\right]$ , and that  $U_{m}^{-}$  is integrable on  $\left[ a,b\right]$ .

A direct application of Theorem 5 gives:

**Theorem 13** It is all as in Remark 12. Here  $\Phi_j$  : $\mathsf{R}^n_{\scriptscriptstyle{+}}$   $\rightarrow$   $\mathsf{R}_{\scriptscriptstyle{+}}$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$   $\!=$   $\!1,...,m$  , are convex functions increasing per coordinate. Then

we that 
$$
L_{jq}(x) > 0
$$
 on  $[a,b]$ .

\n
$$
\epsilon \{1, \ldots, m\} \text{ be fixed. The weight function } u \text{ is chosen so that}
$$
\n
$$
U_{m}^{-}(y) := \left( \prod_{j=1}^{m} \varphi_{j-}(y) \right) \int_{a}^{y} \frac{u(x) \prod_{j=1}^{m} k_{j}^{-}(x, y)}{\prod_{j=1}^{m} L_{jq}(x)} dx < \infty,
$$
\n(57)

\n, *b*, *b*, and that  $U_{m}^{-}$  is integrable on  $[a, b]$ .

\nLet application of Theorem 5 gives:

\nrem 13 It is all as in Remark 12. Here  $\Phi_{j}: \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$ ,  $j = 1, \ldots, m$ , are convex functions increasing per

\nte. Then

\n
$$
\int_{a}^{b} u(x) \prod_{j=1}^{m} \Phi_{j} \left( \frac{\left| \overline{e_{j}^{j,w}}_{j,w_{j},w_{j},b} - f_{j}(x) \right|}{L_{jq}(x)} \right) dx \leq \left( \prod_{j=1}^{m} \int_{a}^{b} \Phi_{j} \left( \frac{\left| \overline{f_{j}(y)} \right|}{\Phi_{j}^{-}(y)} \right) dy \right) \left( \int_{a}^{b} \Phi_{\rho} \left( \frac{\left| \overline{f_{j}(y)} \right|}{\Phi_{j}^{-}(y)} \right) U_{m}^{-}(y) dy \right).
$$
\n(58)

\nTake

We make

 $U_{\infty}(y) := \left[ \prod_{j=1}^{n} \frac{\varphi_{j-}(y)}{\varphi_{j}} \right]_{a}$ <br>  $\int_{y=1}^{\infty} \frac{1}{\varphi_{j}} \frac{\varphi_{j}}{\varphi_{j}}(x)$ <br>  $\mathcal{W} \in [a, b]$ , and that  $U_{\infty}^{-}$  is integrable on  $[a, b]$ <br>
A direct application of Theorem 5 gives:<br> **Theorem 13** It is all a  $\mathcal{L}_{ji} \in C^{N_j}([a,b]), N_j = [\mu_j], \mu_j \notin \mathbb{N};$  $\theta := \max(N_1,...,N_m), \psi \in C^{\theta}([a,b]), \psi$  is increasing with  $\psi'(x) \neq 0$  over  $[a,b]$ . Set [a,b], and that  $U_n^-$  is integrable on [a,b]<br>
lirect application of Theorem 5 gives:<br>
eorem 13 It is all as in Remark 12. Here  $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$ ,  $j = 1,...,m$ , are convex functions in<br>
rate. Then<br>  $\int_a^b u(x) \prod_{j=1}^m \Phi$  $=\left(\frac{1}{\sqrt{1-x}}\right)^{x}$   $\int_{0}^{x} f_{ii}(x)$ dx d  $f_{ji\psi}^{[\cdot]\cdot j]}(x) = \left(\frac{1}{\psi^{'}(x)}\frac{a}{dx}\right) f_{ji}$  $N_{j}$ '  $N_{j}$  $\int_{ii\psi}^{i\pi} f(x) dx = \frac{1}{\psi(x)} \frac{du}{dx}$ J ).  $\overline{ }$  $\setminus$  $\begin{bmatrix} N_j \end{bmatrix}$  $\begin{bmatrix} \mathbf{v} \end{bmatrix}$  $\int_{\psi}^{\pi_{j}} f(x) dx = \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)$   $f_{ji}(x), x \in [a, b]$ . Set n Remark 12. Here  $\Phi_j: \mathbb{R}^n_+ \to \mathbb{R}_+$ ,  $j = 1,...,m$ , are convex functions increasing per<br>  $\frac{m_j, \delta, f_j(x)}{m_j(x)} dx \le \left( \prod_{\substack{j=1 \ j \neq \rho}}^{m} \oint_{\alpha} \left( \frac{f_j(y)}{\phi_j(y)} \right) dy \right) \left( \int_a^{\delta} \Phi_{\rho} \left( \frac{f_j(y)}{\phi_j(y)} \right) U_m(y) dy \right)$  (58)<br>  $\dots, m$ ;  $, \mu$  ;  $, \omega$  ; , =1,..., =1,..., ;  $\left\{ \left\{ \left\langle \mathbf{y}, \mathbf{y}, \mathbf{y}, \mathbf{y}, \mathbf{y} \right\rangle, \left\langle \mathbf{y}, \mathbf{y} \right\rangle \right\} \right\} := \max_{j=1,\ldots,m} \left\{ \left\langle \left[ C D_{\rho_j, \mu_j, \omega_j, a}^{\gamma_j, \mathbf{y}, \mathbf{y}}, \mathbf{y}, \mathbf{y}, \mathbf{y} \right\rangle, \left\langle \mathbf{y}, \mathbf{y} \right\rangle \right\},$  $\left\{ \begin{array}{l} \begin{array}{l} \begin{array}{l} c \end{array} D_{\rho_j,\mu_j,\omega_j,a+1}^{r_j;\psi} \end{array} \end{array} \right. \end{array}$ 8  $\mathcal{A}_{j+}(y) := \left\| {^C}D_{\rho}^{\gamma}{}_{j,\mu}{}_{j,\omega}{}_{j,\mu} + f_j(y) \right\| \ := \max_{j=1} \left\{ \left| {^C}D_{\rho}^{\gamma}{}_{j}{}_{j,\omega}{}_{j,\omega}{}_{j,\mu} + f_j(y) \right\} \right\}$  $i^{\mu}$ j, $\omega_j$ , $a_j$  $\,$  $i = 1, ..., n$  $\left\|f\left(\mathcal{Y}\right)\right\|$  minutes in  $j=1,\ldots,m$ j  $j^{\mu}j^{\mu}j^{\beta}j^{\mu}$  $\,$ j  $\gamma$  ; ;  $\psi$  $\rho$  ; ,  $\mu$  ; ,  $\omega$  $\lambda_{j+}(y) := \left\| {^C}D_{\rho_j,\mu_j,\omega_j,a+}^{y_j;\psi}f_j(y) \right\| := \max_{i=1} \left\{ \left\| {^C}D_{\rho_j,\mu_j,\omega_j,a+}^{y_j;\psi}f_j(y) \right\},\right.$  (59)

and

Type Generalized Fractional Inequalities

\n
$$
\text{George A. Anastassiou}
$$
\n
$$
q\lambda_{j+}(y) := \left\| \overline{C} D_{\rho_j, \mu_j, \omega_j, a+}^{y_j, \psi} f_j(y) \right\|_q := \left( \sum_{i=1}^n \left| \overline{C} D_{\rho_j, \mu_j, \omega_j, a+}^{y_j, \psi} f_j(y) \right|^q \right)^{\frac{1}{q}}, q \ge 1;
$$
\n(60)

\n
$$
\lambda_{j+}
$$
 are continuous functions,  $j = 1, \ldots, m$ . We also have that

 $y\in\!{[a,b]}$ , which all  $_{q}\lambda_{j+}$  are continuous functions,  $\,j$   $\!=$   $\!1,...,m$  . We also have that

$$
0 <_{q} \lambda_{j+}(y) < \infty \text{ in } [a, b], \tag{61}
$$

1

 $j = 1,...,m$ ; where  $1 \le q \le \infty$  is fixed.

Here it is

k <sup>j</sup> x, y:= k <sup>j</sup>x, y = <sup>C</sup> 0, < < , , < , , 1 x y b y x y E x y a y x j j j j j Nj j j N (62)

 $j = 1,...,m$ , and

$$
\sum_{\substack{\rho_j,\mu_j,\omega_j,a+J_j(\nu) \mid q \\ \eta_j,\mu_j,\omega_j,a+J_j(\nu)}}^{\infty} \left\| \left( \sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} \frac{\rho_{\rho_j,\mu_j,\omega_j,a+J_j(\nu)}}{\rho_{j,\mu_j,\omega_j,a+J_j(\nu)}} \right| \right) \right\|, q \ge 1; \tag{60}
$$
\n
$$
\text{Thus, functions, } j = 1, \dots, m. \text{ We also have that}
$$
\n
$$
0 <_{q} \lambda_{j+}(y) < \infty \text{ in } [a, b],
$$
\n
$$
\int_{0}^{1} \left( \frac{y}{\psi(x)} - \psi(y) \right)^{N_j - \mu_j - 1} E_{\rho_j, N_j - \mu_j}^{-\gamma_j} \left[ \omega_j(\psi(x) - \psi(y))^{\rho_j} \right] \, d < y \le x, \tag{62}
$$
\n
$$
x < y < b,
$$
\n
$$
\int_{0}^{C} L_{jq}^*(x) := \int_{a}^{x} \psi'(y) \left( \psi(x) - \psi(y) \right)^{N_j - \mu_j - 1} E_{\rho_j, N_j - \mu_j}^{-\gamma_j} \left[ \omega_j(\psi(x) - \psi(y))^{\rho_j} \right] \, d_{j+}(y) \, dy, \tag{63}
$$

 $\forall x \in [a,b]$ ,  $1 \leq q \leq \infty$ ,  $j = 1,...,m$ .

We have that  ${}^C L^+_{j q} (x) \! > \! 0\,$  on  $\big[a, b\big].$ 

Let  $\rho \in \{1,...,m\}$  be fixed. The weight function  $u$  is chosen so that

$$
\begin{cases}\n & \text{(62)} \\
0, x < y < b,\n\end{cases}
$$
\n
$$
{}^{C}L_{jq}^{+}(x) := \int_{a}^{x} \psi'(y) (\psi(x) - \psi(y))^{N_{j} - \mu_{j} - 1}
$$
\n
$$
E_{\rho_{j}, N_{j} - \mu_{j}}^{-\gamma} [\omega_{j} (\psi(x) - \psi(y))^{\rho_{j}}]_{q} \lambda_{j+}(y) dy,\n\ldots, m
$$
\n
$$
\text{on } [a, b].
$$
\nThe weight function  $u$  is chosen so that\n
$$
{}^{C}U_{m}^{+}(y) := \left(\prod_{j=1}^{m} \lambda_{j+}(y)\right) \int_{y}^{b} \frac{u(x) \prod_{j=1}^{m} \kappa_{j}^{+}(x, y)}{\prod_{j=1}^{m} \kappa_{j}^{+}(x, y)} dx < \infty,\n\tag{64}
$$
\nintegrable on  $[a, b]$ 

 $\forall \, y \in [a,b]$ , and that  $^C U^+_m$  is integrable on  $\big[a,b\big].$ 

A direct application of Theorem 11, see also (6), gives:

**Theorem 15** It is all as in Remark 14. Here  $\Phi_j$  : $\mathsf{R}_+^n$   $\rightarrow$   $\mathsf{R}_+$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$  =  $1,...,m$  , are convex functions increasing per coordinate. Then

**Figure A. Anastassiou**

\n**Example 4.** Anastassiou

\n
$$
\int_{a}^{b} u(x) \prod_{j=1}^{m} \Phi_{j} \left( \frac{\overline{CD_{\rho_{j}, \mu_{j}, \omega_{j}, a+}^{y}}(x)}{\overline{CD_{\rho_{j}, \mu_{j}, \omega_{j}, a+}}^{y}} \right) dx \leq \left( \prod_{j=1}^{m} \int_{a}^{b} \Phi_{j} \left( \frac{\overline{CD_{\rho_{j}, \mu_{j}, \omega_{j}, a+}}^{y}}{\rho_{j} \Phi_{\rho}} \right) dx \right) dy \right) \left( \int_{a}^{b} \Phi_{\rho} \left( \frac{\overline{CD_{\rho_{j}, \mu_{j}, \omega_{j}, a+}}^{y}}{\rho_{j} \Phi_{\rho}} \right) dx \right) dy \right) \tag{65}
$$
\nWe make

\n**Remark 16** Here  $j = 1, ..., m; i = 1, ..., n$ . Let  $\rho_{j}, \mu_{j}, \omega_{j} > 0, \gamma_{j} < 0$ , and  $f_{ji} \in C^{N_{j}}([a, b]), N_{j} = \lceil \mu \rceil, \mu_{j} \notin N;$ 

\n
$$
= \max(N_{1}, ..., N_{m}), \psi \in C^{\theta}([a, b]), \psi \quad \text{is} \quad \text{increasing} \quad \text{with} \quad \psi'(x) \neq 0 \quad \text{over} \quad [a, b]. \quad \text{Set}
$$
\n
$$
\sum_{j=1}^{N_{j}} \int_{x}^{x} f(x) dx = \left( \frac{1}{W(x)} \frac{d}{dx} \right)^{N_{j}} f_{ji}(x), x \in [a, b]. \quad \text{Set}
$$

We make

 $\mathcal{L}_{ji} \in C^{N_j}([a,b]), N_j = [\mu_j], \mu_j \notin \mathbb{N};$  $\theta := \max(N_1, ..., N_m), \psi \in C^{\theta}([a, b]), \psi$  is  $\psi \in C^{\theta}([a,b])$ ,  $\psi$  is increasing with  $\psi'(x) \neq 0$  over  $[a,b]$ . Set A Anastassiou<br>  $\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{\overline{C}D_{j,j',\omega_{j},\sigma_{j},\sigma_{j},f_{j}(x)}}{C_{L_{jq}}(x)} \right) dx \leq \left( \prod_{j=1}^m \int_a^b \Phi_j \left( \frac{\overline{f}_{j,\omega}^{[N_{j}]}(y)}{\sigma_{j,\omega_{j},\sigma_{j},f_{j}(y)}} \right) dy \right) \left( \int_a^b \Phi_j \left( \frac{\overline{f}_{j,\omega}^{[N_{j}]}(y)}{\sigma_{j,\omega_{j},\sigma_{j},f_{j}(y)}} \right)^$  $=\left(\frac{1}{\sqrt{1-x^2}}\right)^{1/y} f_{ii}(x)$ dx d  $f_{ji\psi}^{[\cdot]\cdot j]}(x) = \left(\frac{1}{\psi^{'}(x)}\frac{a}{dx}\right) f_{ji}$  $N_{j}$ '  $N_{j}$  $\int_{ii\psi}^{i+j} f(x) dx = \frac{1}{\psi(x)} \frac{d}{dx}$ J ).  $\overline{ }$  $\setminus$  $\begin{bmatrix} N_j \end{bmatrix}$  $\begin{bmatrix} \mathbf{v} \end{bmatrix}$  $\int_{\psi}^{x_j} f(x) dx = \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right)$   $f_{ji}(x), x \in [a, b]$ . Set Journal of Advances in Applied & Computational Mathematics, 8, 2021<br>  $\frac{m}{\sqrt{n}}(x)$ <br>  $\left| dx \le \left( \prod_{\substack{j=1 \\ j \neq \rho}}^{m} \left| \int_{0}^{b} \Phi_j \left( \frac{\sqrt{\frac{N}{n}} \int_{j \neq \rho}^{N} y}{\sqrt{\frac{1}{n}} \int_{j \neq \rho}} \right) dy \right| dy \right| \left( \int_{0}^{b} \Phi_j \left( \frac{\sqrt{\frac{N}{n}} \int_{\rho$ ; ;  $\,$ j C  $\gamma$ ;; $\psi$ 1,...,*m*; *i* = 1,...,*n*. Let  $\rho_j, \mu_j, \omega_j > 0$ ,  $\gamma_j < 0$ , and  $f_{ji} \in C^N([a, b])$ ,  $N = [a], \mu_j \notin \mathbb{N}$ ;<br>
1,...,*m*; *i* = 1,...,*n*. Let  $\rho_j, \mu_j, \omega_j > 0$ ,  $\gamma_j < 0$ , and  $f_{ji} \in C^N([a, b])$ ,  $N = [a], \mu_j \notin \mathbb{N}$ ;<br>  $C^0([a, b], \psi$  is

 $, \mu$ ;, $\omega$ ; =1,..., =1,...,  $\left\{ \left\{ \left\langle \mathbf{y}, \mathbf{y}, \mathbf{y}, \mathbf{y}, \mathbf{y} \right\rangle \right\} \right\} := \max_{j=1,\ldots,m} \left\{ \left\langle \mathbf{y}, \mathbf{y} \right\rangle \left\langle \mathbf{y}, \mathbf{y}, \mathbf{y}, \mathbf{y} \right\rangle \right\},$  $\left\{\Bigl\vert^cD^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,b-1}\Bigr\vert$ 8  $\mathcal{A}_{j-}(y) := \left\| {^C}D^{y\,j;\Psi}_{\rho_1,\mu_1,\omega_1,b-}f_j(y) \right\| \; := \; \max_{j=1} \left\| {^C}D^{y\,j;\Psi}_{\rho_j,\mu_j,\omega_j,b-}f_j(y) \right\|$  $j^{\mu}$   $j^{\mu}$   $j^{\mu}$  $i = 1, ..., n$  $j \vee j$  max  $j=1,...,m$  $j^{\mu}$ j, $\omega_j$ , $\phi_j$ j  $\rho$  ; ,  $\mu$  ; ,  $\omega$  $\lambda_{j-}(y) := \left\| {^C}D_{\rho_j,\mu_j,\omega_j,b}^{\gamma_j,\psi} - f_j(y) \right\| := \max_{i=1} \left\{ \left\| {^C}D_{\rho_j,\mu_j,\omega_j,b}^{\gamma_j,\psi} - f_j(y) \right\} \right\},$  (66)

and

$$
{}_{q}\lambda_{j-}(y) := \left\| \overrightarrow{C} D_{\rho_{j},\mu_{j},\omega_{j},b-}^{\gamma_{j};\psi} f_{j}(y) \right\|_{q} := \left( \sum_{i=1}^{n} \left| \overrightarrow{C} D_{\rho_{j},\mu_{j},\omega_{j},b-}^{\gamma_{j};\psi} f_{j}(y) \right|^{q} \right)^{\frac{1}{q}}, q \ge 1; \tag{67}
$$

 $y\in\!{[a,b]}$ , which all  $_{q}\lambda_{j-}$  are continuous functions,  $\,j$   $\!=$   $\!1,...,m$  . We also have that

$$
0 <_{q} \lambda_{j-}(y) < \infty \text{ in } [a, b], \tag{68}
$$

 $j = 1,...,m$ ; where  $1 \le q \le \infty$  is fixed.

Here it is

$$
\mathcal{L}_{\mathcal{J}}(y) := \left\| \overline{C} D_{\rho_j, \mu_j, \omega_j, b}^{\gamma_j; \psi} f_j(y) \right\|_{\infty} := \max_{j=1,\dots,m} \left\{ \left[ C D_{\rho_j, \mu_j, \omega_j, b}^{\gamma_j; \psi} f_j(y) \right], \tag{66}
$$
\n
$$
\mathcal{J}_{q} \lambda_{j-}(y) := \left\| \overline{C} D_{\rho_j, \mu_j, \omega_j, b}^{\gamma_j; \psi} f_j(y) \right\|_{q} := \left( \sum_{i=1}^{n} \left[ C D_{\rho_j, \mu_j, \omega_j, b}^{\gamma_j; \psi} f_j(y) \right]^q \right)^{\frac{1}{q}} , q \ge 1;
$$
\n
$$
\mathcal{J}_{j-}(y) := \left\| \overline{C} D_{\rho_j, \mu_j, \omega_j, b}^{\gamma_j; \psi} f_j(y) \right\|_{q} := \left( \sum_{i=1}^{n} \left[ C D_{\rho_j, \mu_j, \omega_j, b}^{\gamma_j; \psi} f_j(y) \right]^q \right)^{\frac{1}{q}} , q \ge 1;
$$
\n
$$
\mathcal{J}_{j-}(y) := \mathcal{J}_{j-}(y) \le \infty \text{ in } [a, b],
$$
\n
$$
\mathcal{J}_{j-}(y) = \infty \text{ in } [a, b],
$$
\n
$$
\mathcal{J}_{j-}(y) = \mathcal{J}_{j
$$

 $j = 1,...,m$ , and

$$
\sum_{j=1}^{n} \sum_{j=1}^{n}
$$

$$
\forall x \in [a, b], 1 \le q \le \infty, j = 1, \dots, m
$$
  
We have that 
$$
{}^{C}L_{jq}^-(x) > 0 \text{ on } [a, b].
$$

Let  $\rho \in \{1,...,m\}$  be fixed. The weight function  $u$  is chosen so that

allized Fractional Inequalities

\nThe weight function 
$$
u
$$
 is chosen so that

\n
$$
{}^{C}U_{m}^{-}(y):=\left(\prod_{j=1}^{m}\lambda_{j-}(y)\right)\int_{a}^{y}\frac{u(x)\prod_{j=1}^{m}{}^{C}k_{j}^{-}(x,y)}{\prod_{j=1}^{m}\frac{C}{L_{jq}^{-}(x)}}dx<\infty,
$$
\n(71)

\nintegrable on  $[a,b]$ .

 $\forall \, y \! \in \! [a,b]$ , and that  $^C U_m^-$  is integrable on  $\big[a,b\big].$ 

A direct application of Theorem 13, see also (7), gives:

**Theorem 17** It is all as in Remark 16. Here  $\Phi_j$  : $\mathsf{R}^n_{\scriptscriptstyle{+}}$   $\rightarrow$   $\mathsf{R}_{\scriptscriptstyle{+}}$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$  =  $1,...,m$  , are convex functions increasing per coordinate. Then

Example 1 Problem

\n1 Problem

\n2Example

\n2Example

\n3Example

\n3Example

\n4Example

\n4 Example

\n5 Example

\n6Example

\n6Example

\n12 The sum of the original matrix 
$$
C \cup \frac{1}{n} \left( \int_{j=1}^{m} \lambda_{j-}(y) \right) \int_{a}^{y} \frac{u(x) \prod_{j=1}^{m} C_{k,j}(x, y)}{\prod_{j=1}^{m} C_{k,j}(x)} dx < \infty,
$$

\n24

\n3Example

\n4 Example

\n12 The sum of the original matrix  $C \cup \frac{1}{n}$  is integrable on  $[a, b]$ .

\n4 Example

\n12 The sum of the original matrix  $C \cup \frac{1}{n}$  is a linear combination of theorem 13, see also (7), gives:

\n12 The sum of the original matrix  $C$  is a linear combination of theorem 13. The sum of the original matrix  $C$  is a linear combination of the form  $C$ 

We make

**Remark 18** Here  $j=1,...,m;$   $i=1,...,n$ . Let  $\mathcal{P}_j, \mu_j, \omega_j > 0$ ,  $\gamma_j < 0$ , and  $f_{ji} \in C([a,b])$ ,  $N_j = \lceil \mu_j \rceil$ ,  $\mu_j \notin \mathsf{N}$ ;  $\theta:=\max(N_1,...,N_m), \, \psi \in C^\theta([a,b]), \, \psi \quad \text{is increasing with} \quad \psi'(x) \neq 0 \quad \text{over} \quad [a,b].$  Here  $0 \leq \beta_j \leq 1$  and  $\xi_j = \mu_j + \beta_j (N_j - \mu_j)$ , We assume that  $\sum_{j}^{RL} D^{y_j (1-p_j) \mu}_{\rho_j, \xi_j, \omega_j, a + f}$ S:<br>  $\begin{aligned}\n&\text{S.}^{\text{in}} \rightarrow \mathbf{R}_{\ast}, \quad j = 1, \dots, m, \text{ are convex functions increasing per} \\
&\text{S.}^{\text{in}}\left\{\frac{\left|f_{j_{\psi}}^{[N_{j}]}(y)\right|}{\sqrt{2\pi}}\right| dy \text{ for } y_{\psi}\left(\frac{\left|f_{j_{\psi}}^{[N_{\rho}]}(y)\right|}{\sqrt{2\pi}}\right) \text{ for } y_{\psi}\left(\frac{\left|f_{j_{\psi}}^{[N_{\rho}]}(y)\right|}{\sqrt{2\pi}}\right) \text{ for } y_{\psi}\left(\frac{\left|f_{$  $i, \xi_j, \omega_j, a$  $^{RL}D^{\gamma_j(1-\beta_j);\psi}_{\rho_i,\xi_i,\omega_i,a+}f_{ji}\in C([a,\bar{b}])$  $^{+}$  $\sum_{\rho_j,\xi_j,\omega_j,a+}^{\gamma_j[1-\beta_j];\psi} f_{ji} \in C([a,b])$ <sub>,</sub>  $j = 1,...,m, i = 1,...,n$ . Set n Remark 16. Here  $\Phi_j: \mathbb{R}^n_+ \to \mathbb{R}_+$ ,  $j = 1,...,m$ , are convex functions increasing per<br>  $\frac{m_j, b \cdot f_j(x)}{m_j(x)} dx \le \left( \prod_{j=0}^m \int_0^b \Phi_j \left( \frac{f_{j\nu}^{[N_j]}(y)}{a^{\lambda_j}-(y)} \right) dy \right) \left( \int_0^b \Phi_j \left( \frac{f_{j\nu}^{[N_p]}(y)}{a^{\lambda_j}-(y)} \right) dy \right$  $,\beta$  ; ;  $, \mu$ ;, $\omega$ ; =1,...,  $,\beta$  ; ; , , ,  $\left\{\n\begin{aligned}\nH \mathbf{D}^{\gamma_{j},\beta_{j};\psi}_{\rho_{j},\mu_{j},\omega_{j},a+\omega_{j}}\n\end{aligned}\n\right.$ 8  $\mathbb{E}_{\mathbb{E}_{\mathbb{E}}[X]}[X] := \left\| H \mathsf{D}^{\gamma_j, \rho_j, \varphi}_{\rho_j, \mu_j, \omega_j, a+} f_j(y) \right\| := \max_{j=1} \left\{ \left\| H \mathsf{D}^{\gamma_j, \rho_j, \varphi}_{\rho_j, \mu_j, \omega_j, a+} f_j(y) \right\} \right\}$  $j^{\mu}j^{\mu}j^{\mu}j^{\mu}$ H  $\left\|f(y)\right\|$   $\left\| \sum_{j=1,\dots,m}^{n} f(y) \right\|$  $i^{\,\prime\,\rho}j$  $i^{\mu_j, \omega_j, a_j}$ H j  $\gamma_i, \beta_i; \psi$  $\mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,a}f_j(y)\| := \max_{i=1} \left\{ \left\| {}^H \mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,a} f_{ji}(y) \right\},\right\},\tag{73}$  $\sum_{j=0}^{n} \frac{1}{L_{jq}(x)} dx \le \left[ \prod_{j\neq j} \int_{0}^{\infty} \Phi_{j} \left( \frac{1}{e^{\lambda_{j-}(y)}} \right) dy \right] \int_{0}^{\infty} \Phi_{\rho} \left( \frac{1}{e^{\lambda_{j-}(y)}} \right) \left( {}^{c}U_{m}^{-}(y)dy \right)$  (72)<br>  $\left[ \prod_{j\neq j} \prod_{i=1}^{n} \ldots n, t_{i} \in \rho_{j}, \mu_{j}, \omega_{j} > 0, \gamma_{j} < 0, \text{ and } f_{j} \in C([a, b]),$ 

=1,...,

 $i = 1, ..., n$ 

and

$$
{}_{q}M_{j+}(y) := \left\| H \mathbf{D}_{\rho_{j}, \mu_{j}, \omega_{j}, a+}^{\gamma_{j}, \beta_{j}; \psi} f_{j}(y) \right\|_{q} := \left( \sum_{i=1}^{n} \left| H \mathbf{D}_{\rho_{j}, \mu_{j}, \omega_{j}, a+}^{\gamma_{j}, \beta_{j}; \psi} f_{j}(y) \right|^{q} \right)^{\frac{1}{q}}, q \ge 1; \tag{74}
$$

 $y\in\!a,b]$ , which all  $_{q}M_{\phantom{a}j+}$  are continuous functions,  $\phantom{j}=1,...,m$  . We also have that

$$
0 \leq_{q} M_{j+}(y) < \infty \text{ in}[a, b],\tag{75}
$$

 $j = 1,...,m$ ; where  $1 \le q \le \infty$  is fixed.

Here it is

Anastassiou

\nJournal of Advances in Applied & Computational Mathematics, 8, 2021

\n
$$
P_{k_j^+}(x, y) := k_j(x, y) = \begin{cases} \psi'(y)(\psi(x) - \psi(y))^{\xi_j - \mu_j - 1} E_{\rho_j, \xi_j - \mu_j}^{-\gamma_j \beta_j} \left[ \omega_j (\psi(x) - \psi(y))^{\rho_j} \right] a < y \leq x, \\ 0, x < y < b, \end{cases} \tag{76}
$$

 $j = 1,...,m$ , and

Journal of Advances in Applied & Computational Mathematics, 8, 2021  
\n
$$
\psi'(y)(\psi(x)-\psi(y))^{\xi_j-\mu_j-1}E_{\rho_j,\xi_j-\mu_j}^{-\gamma_j\beta_j}[\omega_j(\psi(x)-\psi(y))^{\rho_j}]_a < y \le x,
$$
\n
$$
0, x < y < b,
$$
\n
$$
\int_{\rho_j,\xi_j-\mu_j}^{\rho_j} f(x) := \int_a^x \psi'(y)(\psi(x)-\psi(y))^{\xi_j-\mu_j-1} d\psi'(x) d\psi(x) d\psi(x) d\psi(x) d\psi(x) d\psi(x) d\psi(x) d\psi(x).
$$
\n(77)

 $\forall x \in [a, b]$ ,  $1 \le q \le \infty$ .

We have that  ${}^{{}^{{}_{P}}L^{\ast}_{jq}}(x) {>} 0\,$  on  $\big[a,b\big].$ 

Let  $\overline{\rho} \in \{1,...,m\}$  be fixed. The weight function  $u$  is chosen so that

$$
= \n\begin{cases}\n\int_{0, x < y < b,\n\end{cases}\n\tag{76}
$$
\n
$$
{}^{P}L_{jq}^{*}(x) := \int_{a}^{x} \psi'(y)(\psi(x) - \psi(y))^{\varepsilon_{j} - \mu_{j} - 1} \cdot E_{\rho_{j}, \varepsilon_{j} - \mu_{j}}^{-\gamma_{j} \rho_{j}} \left[ \omega_{j} (\psi(x) - \psi(y))^{\rho_{j}} \right]_{a} M_{j+}(y) dy,\n\tag{77}
$$
\n
$$
\int_{0}^{x} \int_{0}^{x} \int_{0}^{x} \int_{0}^{x} \left[ \int_{0}^{x} \psi'(y) \right]_{0}^{x} \left[ \int_{0}^{x} \psi'(y) \right]_{0}^{x} dy \, dy,\n\tag{78}
$$
\n
$$
{}^{P}L_{jq}^{*}(y) := \left( \prod_{j=1}^{m} M_{j+}(y) \right) \int_{y}^{b} \frac{u(x) \prod_{j=1}^{m} {}^{P}L_{j}^{*}(x, y)}{\prod_{j=1}^{m} {}^{P}L_{jq}^{*}(x)} dx < \infty,\n\tag{78}
$$
\n
$$
\int_{0}^{x} \int_{0}^{x} \psi'(y) \left[ \frac{u(x) \int_{0}^{x} \psi'(y) \psi'(y)}{\prod_{j=1}^{m} {}^{P}L_{jq}^{*}(x)} \right]_{0}^{x} dx \leq \infty,\n\tag{78}
$$

 $\forall \, y \in [a,b]$ , and that  $^{\scriptscriptstyle P}U_{\scriptscriptstyle m}^+$  is integrable on  $[a,b]$ .

A direct application of Theorem 11, see also (15), gives:

**Theorem 19** It is all as in Remark 18. Here  $\Phi_j$  : $\mathsf{R}_+^n$   $\rightarrow$   $\mathsf{R}_+$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$  =  $1,...,m$  , are convex functions increasing per coordinate. Then

at <sup>*P*</sup> L<sup>*y*<sub>*n*</sub></sup>(x) > 0 on [a, b].  
\n..., *m*} be fixed. The weight function *u* is chosen so that  
\n
$$
{}^{P}U_{m}^{+}(y):=\left(\prod_{j=1}^{m}M_{j,i}(y)\right)\int_{y}^{u}\frac{u(x)\prod_{j=1}^{m}{}^{P}k_{j}^{+}(x, y)}{\prod_{j=1}^{m}{}^{P}L_{N}^{+}(x)}dx < \infty,
$$
\n(78)  
\nand that <sup>*n*</sup>U<sub>*n*</sub><sup>\*</sup> is integrable on [a, b].  
\nblication of Theorem 11, see also (15), gives:  
\n9 It is all as in Remark 18. Here  $\Phi_{j}: \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$ ,  $j = 1,...,m$ , are convex functions increasing per  
\nen  
\n
$$
\int_{u}^{b}u(x)\prod_{j=1}^{m}\Phi_{j}\left(\frac{m_{D_{j_{j},N_{j},\varpi_{j},\varpi_{j}}(x)}{m_{D_{j},\varpi_{j},\varpi_{j},\varpi_{j}}(x)}\right)dx \leq \left(\prod_{j=0}^{m} \left(\frac{m_{D_{j_{j},\varpi_{j},\varpi_{j},\varpi_{j}}(y)}{m_{D_{j},\varpi_{j},\varpi_{j},\varpi_{j}}(y)}\right)\right)dy
$$
\n(79)  
\n
$$
\left(\int_{u}^{b}\Phi_{j}\left(\frac{m_{D_{j_{j},\varpi_{j},\varpi_{j},\varpi_{j},\varpi_{j}}(y)}{m_{D_{j},\varpi_{j},\varpi_{j},\varpi_{j}}(y)}\right)P_{U_{m}^{+}(y)}(y)dy\right).
$$

We make

**Remark 20** Here  $j=1,...,m;$   $i=1,...,n$ . Let  $\rho_j, \mu_j, \omega_j > 0, \gamma_j < 0$ , and  $f_{ji} \in C([a,b]),$   $N_j = \vert \mu_j \vert, \; \mu_j \notin \mathbb{N};$  $\theta:=\max(N_1,...,N_m), \, \psi \in C^\theta([a,b]), \, \psi$  is increasing with  $\psi^{\prime}(x) \neq 0$  over  $[a,b].$  Here  $0 \leq \beta_j \leq 1$  and  $\zeta_j=\mu_j+\beta_j\big(N_j-\mu_j\big).$  We assume that  $^{RL}D^{r_j(\frac{1-p_j}{p_j})\frac{p_j}{p_j}}_{\rho_j,\xi_j,\omega_j,b}$ George A. Anastassiou<br>  $j, \omega_j > 0, \gamma_j < 0, \text{ and } f_{ji} \in C([a, b]), N_j = |\mu_j|, \mu_j \notin \mathbb{N};$ <br>  $\vdots$  with  $\psi'(x) \neq 0$  over  $[a, b]$ . Here  $0 \leq \beta_j \leq 1$  and<br>  $\psi''(x) \neq 0$  over  $[a, b]$ . Here  $0 \leq \beta_j \leq 1$  and<br>  $\psi''(x) \neq 0$ <br>  $\vdots = \max_{j=1,...,$  $i, \xi_j, \omega_j, b$  $^{RL}D^{\gamma_{j}(1-\beta_{j});\psi}_{\rho_{i},\xi_{i},\omega_{i},b-}f_{ji}\in C([a,$  $\overline{\phantom{a}}$  $\gamma$  ,  $1-\beta$  ;  $|\psi|$  $\sum_{\rho_i,\xi_i,\omega_i,b\in\mathcal{F}}^{\int j^{(1-p)}j^{(2p)}} f_{ji} \in C([a,b]),\,\,j=1,...,m,\,i=1,...,n.$  Set alized Fractional Inequalities<br>
...,*m*; *i* = 1,...,*n*. Let  $\rho_j, \mu_j, \omega_j > 0, \gamma_j < 0$ , and  $f_{ji} \in C([a, b]), N_j = |\mu_j|, \mu_j \notin \mathbb{N}$ ;<br>  $([a, b]), \psi$  is increasing with  $\psi'(x) \neq 0$  over  $[a, b]$ . Here  $0 \leq \beta_j \leq 1$  and<br>
ssume that  ${}^{$  $,\beta$  ; ;  $, \mu$ ;, $\omega$ ; =1,..., =1,...,  $,\beta$  ; ;  $\left\{\left\{\begin{matrix} \beta_j;\psi\\ \mu_j,\omega_j,b-f_j(y)\end{matrix}\right\}\right\}:=\max_{j=1,...,m}\left\{\left[\begin{matrix} H\mathbf{D}_{\rho_j,\mu_j,\omega_j,b-f_{ji}}^{\gamma_j,\beta_j;\psi}(\mathbf{y})\end{matrix}\right],$  $\left\{ \left\{\right.^{\nu} \left\{\right.^{\beta} \right\}^{;\psi}_{\rho_j,\mu_j,\omega_j,b-1} \right.$ 8  $\mathbb{E}_{\mathbb{E}_{\mathbb{E}}[X]}[y] = \|H\mathsf{D}'_{\rho_1,\mu_1,\omega_1,b-}f_j(y)\| \; := \; \max_{i=1} \big\{ \|H\mathsf{D}'_{\rho_1,\mu_1,\omega_1,b-}f_j(y)\| \big\}$  $j^{\mu}j^{\omega}j^{\beta}$ H  $i = 1, ..., n$  $j \vee j$  max  $j=1,...,m$  $i^{\,\prime\prime}$ j  $j^{\mu}$   $j^{\mu}$   $j^{\mu}$ H j  $\gamma$ ;, $\beta$ ;; $\psi$  $\mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b} - f_j(y) \| := \max_{i=1} \left\{ \left| {}^{H} \mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b} - f_{ji}(y) \right| \right\},$  (80) neralized Fractional Inequalities<br>  $\begin{aligned}\n &\text{seorge A Anastasisou} \\
 1, \dots, m; i = 1, \dots, n. \text{ Let } \rho_j, \mu_j, \omega_j > 0, \ \gamma_j < 0, \text{ and } f_{ji} \in C([a, b]), \ N_j = |\mu_j|, \ \mu_j \notin \mathbb{N}; \\
 &C^{\theta}([a, b]), \ \psi \quad \text{is increasing with} \quad \psi'(x) \neq 0 \quad \text{over } [a, b]. \text{ Here } 0 \leq \beta_j \leq 1 \text{ and } \\
 &\text{e assume that } \$ 

and

$$
{}_{q}M_{j-}(y) := \left\| H \mathbf{D}_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{\gamma_{j}, \beta_{j}, \psi} - f_{j}(y) \right\|_{q} := \left( \sum_{i=1}^{n} \left| H \mathbf{D}_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{\gamma_{j}, \beta_{j}, \psi} - f_{j}(y) \right|^{q} \right)^{\frac{1}{q}}, q \ge 1; \tag{81}
$$

 $y\in\!a,b]$ , which all  $_{q}M$   $_{j-}$  are continuous functions,  $\ j=1,...,m$  . We also have that

$$
0 \leq_{q} M_{j-}(y) < \infty \text{ in}[a, b], \tag{82}
$$

 $j = 1,...,m$ ; where  $1 \le q \le \infty$  is fixed.

Here it is

$$
\sum_{\alpha} M_{j-}(y) := \left\| \frac{\partial D_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{y - y - f_{j}}(y)}{\partial_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{y - f_{j}}(y)} \right\|_{\infty} := \max_{j=1,...,n} \left\| \frac{\partial D_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{y - y - f_{j}}(y)}{\partial_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{y - f_{j}}(y)} \right\|_{\infty}
$$
\n(80)  
\n
$$
{}_{q}M_{j-}(y) := \left\| \frac{\partial D_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{y - y - f_{j}}(y)}{\partial_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{y - f_{j}}(y)} \right\|_{\infty} := \left( \sum_{i=1}^{n} \left\| D_{\rho_{j}, \mu_{j}, \omega_{j}, b}^{y - y - f_{j}}(y) \right\|_{\infty}^{q} \right) \frac{1}{q}, q \ge 1;
$$
\n(81)  
\nwhich all  ${}_{q}M_{j-}$  are continuous functions,  $j = 1,...,m$ . We also have that  
\n
$$
0 <_{q}M_{j-}(y) < \infty
$$
 in [a, b].  
\n11, where  $1 \le q \le \infty$  is fixed.  
\nis  
\n
$$
{}^{p}K_{j}(x, y) := k_{j}(x, y) = \begin{cases} \psi'(y)(\psi(y) - \psi(x))^{e_{j} - \mu_{j} - 1} E_{\rho_{j}, \varepsilon_{j} - \mu_{j}}^{-y - \mu_{j} - 1} \left[ \omega_{j}(\psi(y) - \psi(x))^{p_{j}} \right] x \le y < b, \\ 0, a < y < x, \end{cases}
$$
\n(83)  
\n17, and  
\n
$$
{}^{p}L_{jq}(x) := \int_{x}^{b} \psi'(y)(\psi(y) - \psi(x))^{e_{j} - \mu_{j} - 1} E_{\rho_{j}, \varepsilon_{j} - \mu_{j}}^{-y - \mu_{j} - 1} \left[ \omega_{j}(\psi(y) - \psi(x))^{p_{j}} \right]_{\infty} M_{j-}(y) dy,
$$

 $j = 1,...,m$ , and

$$
{}^{P}L_{jq}^{-}(x) := \int_{x}^{b} \psi^{'}(y) (\psi(y) - \psi(x))^{\xi_{j} - \mu_{j} - 1} \quad E_{\rho_{j}, \xi_{j} - \mu_{j}}^{-\gamma_{j} \beta_{j}} \left[ \omega_{j} (\psi(y) - \psi(x))^{\rho_{j}} \right]_{q} M_{j-}(y) dy,
$$
\n(84)

 $\forall x \in [a, b]$ ,  $1 \le q \le \infty$ .

We have that  ${}^{{}^{{}_{P}}L_{jq}^{-}}(x) {>} 0\,$  on  $\big[a, b\big].$ 

Let  $\overline{\rho} \in \{1,...,m\}$  be fixed. The weight function  $u$  is chosen so that

$$
y = \begin{cases} \psi'(y)(\psi(y) - \psi(x))^{\varepsilon_{j} - \mu_{j} - 1} E_{\rho_{j}, \varepsilon_{j} - \mu_{j}}^{-\gamma_{j} \beta_{j}} [\omega_{j}(\psi(y) - \psi(x))^{\rho_{j}}] x \leq y < b, \\ 0, a < y < x, \end{cases}
$$
\n
$$
(83)
$$
\n
$$
y'(y)(\psi(y) - \psi(x))^{\varepsilon_{j} - \mu_{j} - 1} E_{\rho_{j}, \varepsilon_{j} - \mu_{j}}^{-\gamma_{j} \beta_{j}} [\omega_{j}(\psi(y) - \psi(x))^{\rho_{j}}]_{q} M_{j-}(y) dy, \qquad (84)
$$
\n
$$
y \text{ on } [a, b].
$$
\n
$$
\text{The weight function } u \text{ is chosen so that}
$$
\n
$$
f^{P}U_{m}^{-}(y) := \left( \prod_{j=1}^{m} M_{j-}(y) \right) \int_{a}^{y} \frac{u(x) \prod_{j=1}^{m} f_{k_{j}}(x, y)}{\prod_{j=1}^{m} f_{k_{j}}(x)} dx < \infty, \qquad (85)
$$
\n
$$
\text{integrable on } [a, b].
$$

 $\forall \, \, y \! \in \! \! [a,b]$ , and that  $^{\mathit{P}}U_{\mathit{m}}^-$  is integrable on  $\big[a,b\big]$ .

A direct application of Theorem 13, see also (16), gives:

**Theorem 21** It is all as in Remark 20. Here  $\Phi_j$  : $\mathsf{R}_+^n$   $\rightarrow$   $\mathsf{R}_+$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$  =  $1,...,m$  , are convex functions increasing per coordinate. Then

 dx L x f x u x jq P j j j b j j j H j m j b a , ; , , , =1 <sup>D</sup> dy <sup>M</sup> <sup>y</sup> D f y q j j j j b j j j RL j b a m j j 1 ; , , , =1 (86) . 1 ; , , , <sup>U</sup> <sup>y</sup> dy <sup>M</sup> <sup>y</sup> D f y m P q j j b j j j RL b a := <sup>=</sup> , , 

We make

**Remark 22** The basic background here is as in Remark 10. Also  $_q\varphi_{j+}(y)$ ,  $1\leq q\leq\infty$  ,  $y\in[a,b]$  is as in (45), (46), (47);  $k^+_j(x,y)$  is as (48) and  $L^+_{jq}(x)$  as in (49), where  $\,x,y\in\lbrack a,b\rbrack.$  Here it is

$$
K_j^+(x) := K_j(x) = (\psi(x) - \psi(a))^{\mu_j} E_{\rho_j, \mu_j+1}^{\gamma_j} [\omega_j(\psi(x) - \psi(a))^{\rho_j}]
$$
\n(87)

 $\forall x \in [a,b], j = 1,...,m$  . Indeed it is

The basic background here is as in Remark 10. Also 
$$
{}_{q}\varphi_{j+}(y)
$$
,  $1 \leq q \leq \infty$ ,  $y \in [a,b]$  is as in (45), (46),  
\ns as (48) and  $L_{jq}^{+}(x)$  as in (49), where  $x, y \in [a,b]$ . Here it is  
\n
$$
K_{j}^{+}(x) := K_{j}(x) = (\psi(x) - \psi(a))^{\omega_{j}} E_{\rho_{j}, \mu_{j}+1}^{y_{j}} [\omega_{j}(\psi(x) - \psi(a))^{\rho_{j}}]
$$
\n
$$
= 1,...,m
$$
. Indeed it is  
\n
$$
\frac{k_{j}^{+}(x,y)}{K_{j}^{+}(x)} = \left(\chi_{(a,x]}(y)\psi^{+}(y)\frac{(\psi(x) - \psi(y))^{\mu_{j}-1}}{(\psi(x) - \psi(a))^{\rho_{j}}}\right) \left(\frac{E_{\rho_{j}, \mu_{j}}^{y_{j}} [\omega_{j}(\psi(x) - \psi(y))^{\rho_{j}}]}{E_{\rho_{j}, \mu_{j}+1}^{y_{j}} [\omega_{j}(\psi(x) - \psi(a))^{\rho_{j}}]}\right),
$$
\n(88)  
\n $j = 1,...,m; \chi$  is the characteristic function.  
\n $W_{j+}$  on [a,b], with appropriate choice of weight function *u*, by  
\n
$$
{}_{q}W_{j+}(y) := {}_{q}\varphi_{j+}(y) \left(\int_{y}^{b} \frac{u(x)k_{j}^{+}(x,y)}{K_{j}^{+}(x)} dx\right) < \infty,
$$
\n(89)  
\nand that  ${}_{q}W_{j+}$  is integrable on [a,b];  $j = 1,...,m$ .

 $\forall x, y \in [a,b], j = 1,...,m; \chi$  is the characteristic function.

We define  $_{q}W_{j+}$  on  $\left[ a,b\right]$ , with appropiate choice of weight function  $u$  , by

$$
{}_{q}W_{j+}(y):= {}_{q}\varphi_{j+}(y)\bigg(\int_{y}^{b}\frac{u(x)k_{j}^{+}(x,y)}{K_{j}^{+}(x)}dx\bigg)<\infty,
$$
\n(89)

 $\forall y \in [a,b]$ , and that  $_{q}W_{j+}$  is integrable on  $[a,b]$ ;  $j=1,...,m.$ 

A direct application of Theorem 6, see also (2), follows:

**Theorem 23** It is all as in Remark 22. Let  $p_j > 1$ :  $\sum_{i=1}^{m} \frac{1}{n_i} = 1$ .  $=$ 1  $P_j$ m j  $p_j > 1$  :  $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$ . Let the functions  $\Phi_j : \mathsf{R}_+^n \to \mathsf{R}_+$  $j:$   $\mathsf{R}^n_+$   $\to$   $\mathsf{R}^n_+$ ,  $j=1,...,m$  , be

convex and increasing per coordinate. Then

day Type Generalized Fractional Inequalities  
\n
$$
\int_{a}^{b} u(x) \prod_{j=1}^{m} \Phi_{j} \left( \frac{\overline{e^{y_{j}, w_{j}, a_{j}, a_{j}, f_{j}(x)}}}{L_{jq}^{+}(x)} \right) dx \leq \prod_{j=1}^{m} \left( \int_{a_{q}}^{b} W_{j+}(y) \Phi_{j} \left( \frac{\overline{f_{j}(y)}}{\phi_{j+}(y)} \right)^{p_{j}} dy \right)^{\frac{1}{p_{j}}}. \qquad (90)
$$
\ne basic background here is as in Remark 12. Also  ${}_{q} \phi_{j-}(y)$ ,  $1 \leq q \leq \infty$ ,  $y \in [a, b]$  is as in (52), (53),  
\n5(55) and  $L_{jq}(x)$  as in (56), where  $x, y \in [a, b]$ . Here it is  
\n
$$
K_{j}^{-}(x) := K_{j}(x) = (w(b) - w(x))^{u_{j}} E_{\rho_{j}, u_{j}+1}^{y_{j}} \left[ \omega_{j} (w(b) - w(x))^{\rho_{j}} \right]
$$
\n
$$
...
$$
*m*. Indeed it is  
\n
$$
x, y \geq \left( \chi_{[x,b)}(y)w'(y) \frac{(w(y) - w(x))^{\mu_{j}-1}}{(w(b) - w(x))^{y_{j}}} \right) \left( \frac{E_{\rho_{j}, u_{j}+1}^{y_{j}} [\omega_{j} (w(y) - w(x))^{\rho_{j}}]}{E_{\rho_{j}, u_{j}+1}^{y_{j}} [\omega_{j} (w(b) - w(x))^{\rho_{j}}]} \right), \qquad (92)
$$

We make

**Remark 24** The basic background here is as in Remark 12. Also  $_q\varphi_j(y)$ ,  $1\leq q\leq\infty$  ,  $y\in[a,b]$  is as in (52), (53), (54);  $k_{j}^{-}(x,y)$  is as (55) and  $L_{jq}^{-}(x)$  as in (56), where  $\,x,y\in\!{[a,b]}.$  Here it is

$$
K_j^-(x) := K_j(x) = (\psi(b) - \psi(x))^{\mu_j} E_{\rho_j, \mu_j+1}^{y_j} [\omega_j(\psi(b) - \psi(x))^{\rho_j}]
$$
\n(91)

 $\forall x \in [a, b], j = 1,...,m$  . Indeed it is

**4 The basic background here is as in Remark 12. Also** 
$$
{}_{q}\varphi_{j-}(y), 1 \leq q \leq \infty, y \in [a, b]
$$
 is as in (52), (53), is as (55) and  $L_{jq}(x)$  as in (56), where  $x, y \in [a, b]$ . Here it is

\n
$$
K_{j}^{-}(x) := K_{j}(x) = (\psi(b) - \psi(x))^{\mu_{j}} E_{\rho_{j}, \mu_{j}+1}^{Y_{j}} [\omega_{j}(\psi(b) - \psi(x))^{\rho_{j}}]
$$
\n(91)

\n
$$
j = 1, \ldots, m \text{ Indeed it is}
$$
\n
$$
\frac{k_{j}^{-}(x, y)}{K_{j}^{-}(x)} = \left( \chi_{[x, b)}(y) \psi'(y) \frac{(\psi(y) - \psi(x))^{\mu_{j-1}}}{(\psi(b) - \psi(x))^{\mu_{j}}} \right) \left( \frac{E_{\rho_{j}, \mu_{j}}^{y_{j}} [\omega_{j}(\psi(y) - \psi(x))^{\rho_{j}}]}{E_{\rho_{j}, \mu_{j}+1}^{y_{j}} [\omega_{j}(\psi(b) - \psi(x))^{\rho_{j}}]} \right), \qquad (92)
$$
\n
$$
j = 1, \ldots, m
$$
\n
$$
j = 1, \ldots, m
$$
\n
$$
{}_{q}W_{j-}
$$
 on  $[a, b]$ , with appropriate choice of weight function  $u$ , by\n
$$
{}_{q}W_{j-}(y) := {}_{q}\varphi_{j-}(y) \left( \int_{a}^{y} \frac{u(x)k_{j}^{-}(x, y)}{K_{j}^{-}(x)} dx \right) < \infty, \qquad (93)
$$
\nand that

\n
$$
{}_{q}W_{j-}
$$
 is integrable on  $[a, b]$ ;  $j = 1, \ldots, m$ .\n**pllication of Theorem 6. see also (3). follows:**

 $\forall x, y \in [a, b], j = 1,...,m$ 

We define  $_{q}W_{j-}$  on  $\left[ a,b\right]$ , with appropiate choice of weight function  $u$  , by

$$
{}_{q}W_{j-}(y):= {}_{q}\varphi_{j-}(y)\left(\int_{a}^{y}\frac{u(x)k_{j}^{-}(x,y)}{K_{j}^{-}(x)}dx\right)<\infty,
$$
\n(93)

 $\forall y \in [a,b]$ , and that  $_{q}W_{j}$  is integrable on  $[a,b]$ ;  $j = 1,...,m$ .

A direct application of Theorem 6, see also (3), follows:

**Theorem 25** It is all as in Remark 24. Let  $p_j > 1$ :  $\sum_{i=1}^{m} \frac{1}{n_i} = 1$ .  $=$ 1  $P_j$ m j  $p_j > 1$  :  $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$ . Let the functions  $\Phi_j : \mathsf{R}_+^n \to \mathsf{R}_+$  $j_i: \mathsf{R}_+^n \to \mathsf{R}_+$ ,  $j=1,...,m$ , be convex and increasing per coordinate. Then

$$
\left(\begin{array}{c} p_{j}u_{j+1}(-y) \leftrightarrow (y_{j} - y_{j+1}(-y) \leftrightarrow (y_{j} - y_{j+1})\right) \\ \text{or } [a,b], \text{ with appropriate choice of weight function } u, \text{ by} \\ \frac{1}{q}W_{j-}(y):=_{q} \varphi_{j-}(y) \left(\int_{a}^{y} \frac{u(x)k_{j}^{-}(x,y)}{K_{j}^{-}(x)} dx\right) < \infty, \qquad (93)
$$
\n
$$
\text{that }_{q}W_{j-} \text{ is integrable on } [a,b]; \ j=1,...,m.
$$
\n
$$
\text{ation of Theorem 6, see also (3), follows:}
$$
\n
$$
\text{It is all as in Remark 24. Let } p_{j} > 1: \sum_{j=1}^{m} \frac{1}{p_{j}} = 1. \text{ Let the functions } \Phi_{j}: \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}, \ j=1,...,m, \text{ be} \text{using per coordinate. Then}
$$
\n
$$
\int_{a}^{b} u(x) \prod_{j=1}^{m} \Phi_{j} \left(\frac{e^{y_{j}w_{j}}\psi_{j} \otimes y_{j} - f_{j}(x)}{L_{j}(\mathbf{x})}\right) dx \leq \prod_{j=1}^{m} \left(\int_{a}^{b} W_{j-}(y) \Phi_{j} \left(\frac{\overline{f_{j}(y)}}{\overline{f_{j}(y)}}\right)^{p_{j}} dy\right)^{\frac{1}{p_{j}}}.
$$
\n
$$
\tag{94}
$$

We need

**Remark 26** The basic background here is as in Remark 14. Also  $_{q}\lambda_{j+}(y)$ ,  $1\leq q\leq\infty$ ,  $y\in[a,b]$  is as in (59), (60), (61);  ${}^C k_j^+(x,y)$  is as (62) and  ${}^C L_{jq}^+(x)$  as in (63), where  $\,x,y\in\lbrack a,b\rbrack.$  Here it is

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\n
$$
{}^{C}K_{j}^{+}(x):=K_{j}(x)=(\psi(x)-\psi(a))^{N_{j}-\mu_{j}}E_{\rho_{j},N_{j}-\mu_{j}+1}^{-\gamma_{j}}[\omega_{j}(\psi(x)-\psi(a))^{\rho_{j}}]
$$
\n
$$
\dots, m \text{ Indeed it is}
$$
\n(95)

 $\forall x \in [a, b], j = 1,...,m$  . Indeed it is

stassiou  
\n
$$
^{C}K_{j}^{+}(x) := K_{j}(x) = (\psi(x) - \psi(a))^{N_{j} - \mu_{j}} E_{\rho_{j}, N_{j} - \mu_{j}+1}^{-\gamma_{j}} [\omega_{j}(\psi(x) - \psi(a))^{\rho_{j}}]
$$
\n(95)  
\n9],  $j = 1,...,m$ . Indeed it is  
\n
$$
\frac{^{C}k_{j}^{+}(x, y)}{^{C}K_{j}^{+}(x)} = \left(\chi_{(a,x]}(y)\psi'(y)\frac{(\psi(x) - \psi(y))^{N_{j} - \mu_{j} - 1}}{(\psi(x) - \psi(a))^{N_{j} - \mu_{j}}}\right) \left(\frac{E_{\rho_{j}, N_{j} - \mu_{j}}^{-\gamma_{j}} [\omega_{j}(\psi(x) - \psi(y))^{\rho_{j}}]}{E_{\rho_{j}, N_{j} - \mu_{j}+1}^{-\gamma_{j}} [\omega_{j}(\psi(x) - \psi(y))^{\rho_{j}}]}\right),
$$
\n(96)  
\n1, b],  $j = 1,...,m$   
\n1, b],  $j = 1,...,m$   
\n1, c],  $\int_{a}^{C}W_{j+}$  on [a, b], with appropriate choice of weight function u, by  
\n
$$
\int_{a}^{C}W_{j+}(y) :=_{q} \lambda_{j+}(y) \left(\int_{y}^{b} \frac{u(x)^{C}k_{j}^{+}(x, y)}{^{C}K_{j}^{+}(x)} dx\right) < \infty,
$$
\n(97)  
\n1, and that  $\int_{q}^{C}W_{j+}$  is integrable on [a, b];  $j = 1,...,m$ .  
\n(97)  
\n1, and that  $\int_{q}^{C}W_{j+}$  is integrable on [a, b];  $j = 1,...,m$ .

 $\forall x, y \in [a, b], j = 1,...,m$ 

We define  $\frac{c}{q}W_{j+}$  on  $\big[a,b\big],$  with appropiate choice of weight function  $\,u$  , by

$$
{}_{q}^{C}W_{j+}(y):= {}_{q}\lambda_{j+}(y)\left(\int_{y}^{b}\frac{u(x)^{C}k_{j}^{+}(x,y)}{{}_{C}K_{j}^{+}(x)}dx\right)<\infty,
$$
\n(97)

 $\forall y \in [a,b]$ , and that  $\frac{c}{q}W_{j+}$  is integrable on  $[a,b]$ ;  $j = 1,...,m$ .

A direct application of Theorem 23, see also (6), follows:

**Theorem 27** It is all as in Remark 26. Let  $p_j > 1$ :  $\sum_{i=1}^{m} \frac{1}{n_i} = 1$ .  $=$ 1  $P_j$ m j  $p_j > 1$  :  $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$ . Let the functions  $\Phi_j : \mathsf{R}_+^n \to \mathsf{R}_+$  $j:$   $\mathsf{R}^n_+$   $\to$   $\mathsf{R}^n_+$ ,  $j=1,...,m$  , be convex and increasing per coordinate. Then

$$
j = 1,...,m
$$
\n
$$
V_{j+}
$$
 on [a,b], with appropriate choice of weight function *u*, by\n
$$
\frac{c}{v}W_{j+}(y):=_{v} \lambda_{j+}(y) \left( \int_{y}^{u} \frac{u(x)^{v} k_{j}^{+}(x, y)}{c^{v} k_{j}^{+}(x)} dx \right) < \infty,
$$
\n
$$
V_{j+}
$$
 is integrable on [a,b];  $j = 1,...,m$ .\n
$$
j = 1,...,m.
$$
\n
$$
V_{j+}
$$
 is integrable on [a,b];  $j = 1,...,m$ .\n
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V_{j+}
$$
 is integrable on [a,b];  $j = 1,...,m$ .\n
$$
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$$
 is an  $n = 23$ , see also (6), follows:\n
$$
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We need

**Remark 28** The basic background here is as in Remark 16. Also  $_q\lambda_{j-}(y)$ ,  $1\leq q\leq\infty$ ,  $y\in[a,b]$  is as in (66), (67), (68);  ${}^Ck_j^-(x,y)$  is as (69) and  ${}^CL_{jq}^-(x)$  as in (70), where  $\,x,y\in\lbrack a,b\rbrack.$  Here it is

$$
{}^{C}K_{j}^{-}(x):=K_{j}(x)=(\psi(b)-\psi(x))^{N_{j}-\mu_{j}}E_{\rho_{j},N_{j}-\mu_{j}+1}^{-\gamma_{j}}[\omega_{j}(\psi(b)-\psi(x))^{\rho_{j}}]
$$
\n(99)

 $\forall x \in [a,b], j = 1,...,m$  . Indeed it is

$$
\frac{{}^{c}k_{j}^{-}(x,y)}{{}^{c}K_{j}^{-}(x)}=\left(\chi_{[x,b)}(y)\psi^{'}(y)\frac{(\psi(y)-\psi(x))^{N_{j}-\mu_{j}-1}}{(\psi(b)-\psi(x))^{N_{j}-\mu_{j}}}\right)\left(\frac{E_{\rho_{j},N_{j}-\mu_{j}}^{-\gamma_{j}}\left[\omega_{j}(\psi(y)-\psi(x))^{\rho_{j}}\right]}{E_{\rho_{j},N_{j}-\mu_{j}+1}^{-\gamma_{j}}\left[\omega_{j}(\psi(b)-\psi(x))^{\rho_{j}}\right]}\right),\qquad(100)
$$

 $\forall x, y \in [a, b], j = 1,...,m.$ 

We define  $\frac{c}{q}W_{j-}$  on  $\big[a,b\big],$  with appropiate choice of weight function  $\,u$  , by

 <sup>&</sup>lt; , , := dx K x <sup>u</sup> <sup>x</sup> <sup>k</sup> <sup>x</sup> <sup>y</sup> <sup>W</sup> <sup>y</sup> <sup>y</sup> j C j C y a j q j C <sup>q</sup> (101)

 $\forall \;\; y \in \big[a,b\big],$  and that  $\frac{c}{q}W_{j-}$  is integrable on  $\big[a,b\big];\; j=1,...,m.$ 

A direct application of Theorem 25, see also (7), follows:

**Theorem 29** It is all as in Remark 28. Let  $p_j > 1$ :  $\sum_{i=1}^{m} \frac{1}{n_i} = 1$ .  $=$ 1  $P_j$ m j  $p_j > 1$  :  $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$ . Let the functions  $\Phi_j : \mathsf{R}_+^n \to \mathsf{R}_+$  $j:$   $\mathsf{R}^n_+$   $\to$   $\mathsf{R}^n_+$ ,  $j=1,...,m$  , be convex and increasing per coordinate. Then

car Hardy type generalized Fractional inequalities  
\n<sup>c</sup><sub>q</sub> 
$$
W_{j-}
$$
 on [a,b], with appropriate choice of weight function *u*, by  
\n<sup>c</sup><sub>q</sub>  $W_{j-}$  (y):=<sub>q</sub>  $\lambda_{j-}$ (y)  $\left(\int_{a}^{x} \frac{u(x)^{C}k_{j}(x,y)}{C^{C}k_{j}(x,y)}dx\right) < \infty$ , (101)  
\nand that  $\frac{c}{q}W_{j-}$  is integrable on [a,b]; *j* = 1,...,*m*.  
\npplication of Theorem 25, see also (7), follows:  
\n29 It is all as in Remark 28. Let  $p_{j} > 1$ :  $\sum_{j=1}^{m} \frac{1}{p_{j}} = 1$ . Let the functions  $\Phi_{j}: \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$ , *j* = 1,...,*m*, be  
\nincreasing per coordinate. Then  
\n
$$
\int_{a}^{b} u(x) \prod_{j=1}^{m} \Phi_{j} \left( \frac{\overline{c_{D_{j}, u_{j}, a_{j}, b-} f_{j}(x)}}{C_{\overline{L}_{jq}}(x)} \right) dx \leq \prod_{j=1}^{m} \int_{a_{q}}^{a_{q}} W_{j-}
$$
(y)  $\Phi_{j} \left( \frac{\overline{f_{j,v}^{(y,j)}}(y)}{q\lambda_{j-}$  (102)  
\n30 The basic background here is as in Remark 18. Also  ${}_{q}M_{j+}$ (y),  $1 \leq q \leq \infty$ ,  $y \in [a,b]$  is as in (73),  
\n $\int_{a}^{+}(x, y)$  is as (76) and  ${}^{n}L_{jq}^{+}(x)$  as in (77), where  $x, y \in [a,b]$ . Here it is  
\n $P_{K_{j}^{+}}(x):= K_{j}(x) = (\psi(x)-\psi(a))^{g_{j}-u_{j-1}} \sum_{j=1}^{n} \sum_{j=1}^{n} \psi_{j} \left( \psi(x)-\psi(a) \right)^{p} / \int_{a}^{b} \left( \frac{\overline{c_{D_{j}^{+}}(x)}(x)}{\overline{c_{D_{j}$ 

We need

**Remark 30** The basic background here is as in Remark 18. Also  $_{q}M_{j+}(y)$ ,  $1 \le q \le \infty$ ,  $y \in [a,b]$  is as in (73), (74), (75);  ${^P}k_j^+(x,y)$  is as (76) and  ${^P}L_{jq}^+(x)$  as in (77), where  $\,x,y\in\!{[a,b]}.$  Here it is

$$
{}^{P}K_{j}^{+}(x):=K_{j}(x)-(\psi(x)-\psi(a))^{\xi_{j}-\mu_{j}}E_{\rho_{j},\xi_{j}-\mu_{j}+1}^{-\gamma_{j}\beta_{j}}[\omega_{j}(\psi(x)-\psi(a))^{\rho_{j}}].
$$
\n(103)

 $\forall x \in [a,b], j = 1,...,m$  . Indeed it is

rk 30 The basic background here is as in Remark 18. Also 
$$
{}_{q}M_{j+}(y)
$$
,  $1 \leq q \leq \infty$ ,  $y \in [a,b]$  is as in (73),  
\n ${}^{p}k_{j}^{+}(x,y)$  is as (76) and  ${}^{p}L_{jq}^{+}(x)$  as in (77), where  $x, y \in [a,b]$ . Here it is  
\n ${}^{p}K_{j}^{+}(x):=K_{j}(x)=(\psi(x)-\psi(a))^{\xi_{j}-\mu_{j}}E_{\rho_{j},\xi_{j}-\mu_{j}+1}^{-\gamma_{j}\beta_{j}}[\omega_{j}(\psi(x)-\psi(a))^{\rho_{j}}]$  (103)  
\nb],  $j=1,...,m$ . Indeed it is  
\n $\frac{{}^{p}k_{j}^{+}(x,y)}{{}^{p}K_{j}^{+}(x)}=\left(\chi_{(a,x)}(y)\psi'(y)\frac{(\psi(x)-\psi(y))^{\xi_{j}-\mu_{j}-1}}{(\psi(x)-\psi(a))^{\xi_{j}-\mu_{j}}}\right)\left(\frac{E_{\rho_{j},\xi_{j}-\mu_{j}}^{-\gamma_{j}\beta_{j}}[\omega_{j}(\psi(x)-\psi(y))^{\rho_{j}}]}{E_{\rho_{j},\xi_{j}-\mu_{j}+1}^{-\gamma_{j}\beta_{j}}[\omega_{j}(\psi(x)-\psi(a))^{\rho_{j}}]}\right),$  (104)  
\n*a,b*],  $j=1,...,m$ .  
\n  
\nfinite  $\frac{{}^{p}W_{j+}}$  on [*a,b*], with appropriate choice of weight function *u*,  
\n $\frac{{}^{p}W_{j+}}{{}^{q}W_{j+}}(y):=\frac{{}_{q}M_{j+}}(y)\left(\int_{y}^{b}\frac{u(x)^{p}k_{j}^{+}(x,y)}{{}^{p}K_{j}^{+}(x)}dx\right)<\infty,$   
\n*b*], and that  $\frac{{}^{p}W_{j+}}$  is integrable on [*a,b*];  $j=1,...,m$ .  
\n  
\n $t$  andization of Theorem 23, see also (15) follows:

 $\forall x, y \in [a, b], j = 1,...,m$ 

We define  ${^{P}_{q}}W_{j+1}$ 

$$
{}_{q}^{P}W_{j+}(y):= {}_{q}M_{j+}(y)\left(\int_{y}^{b}\frac{u(x)^{P}k_{j}^{+}(x,y)}{{}_{P}K_{j}^{+}(x)}dx\right)<\infty,
$$
\n(105)

 $\forall \; y \in \big[a,b\big],$  and that  $\frac{P}{q}W_{j+}$  is integrable on  $\big[a,b\big];\; j=1,...,m.$ 

A direct application of Theorem 23, see also (15), follows:

**Theorem 31** It is all as in Remark 30. Here  $\Phi_j$  : $\mathsf{R}_+^n$   $\rightarrow$   $\mathsf{R}_+$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$  =  $1,...,m$  , are convex functions increasing per coordinate. Then

astasiou  
\npartial of Advances in Applied & Computational Mathematics, 8, 2021  
\n**em 31** It is all as in Remark 30. Here 
$$
\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+
$$
,  $j = 1,...,m$ , are convex functions increasing per  
\ne. Then  
\n
$$
\int_a^b u(x) \prod_{j=1}^m \Phi_j \left( \frac{m \sum_{j=1}^m y_{j+1}^m \psi_{j+1,j}(x)}{p_{j+1}^m \psi_{j+1,j}(x)} \right) dx \le \prod_{j=1}^m \left( \int_a^b w_{j+1}^m \psi_{j+1}(y) \Phi_j \left( \frac{w \sum_{j=1}^m y_{j+1}^m \psi_{j+1,j}(y)}{q M_{j+1}(y)} \right)^{p_j} dy \right)^{-1} dy
$$
\n
$$
= \left( \mathbb{R} \mathbf{B} \mathbf{B} \mathbf{B} \mathbf{B} \mathbf{C} \mathbf{D} \mathbf{C} \mathbf{D} \math
$$

We need

**Remark 32** The basic background here is as in Remark 20. Also  $_{q}M$   $_{j-}(y)$ ,  $1\leq q\leq\infty$ ,  $y\in\lbrack a,b]$  is as in (80), (81), (82);  ${}^PK_j^-(x,y)$  is as in (83) and  ${}^PL_{jq}^-(x)$  as in (84), where  $\,x,y\in\lbrack a,b\rbrack.$  Here it is

$$
{}^{P}K_{j}^{-}(x):=K_{j}(x)=(\psi(b)-\psi(x))^{\xi_{j}-\mu_{j}}E_{\rho_{j},\xi_{j}-\mu_{j}+1}^{-\gamma_{j}\beta_{j}}[\omega_{j}(\psi(b)-\psi(x))^{\rho_{j}}]
$$
\n(107)

 $\forall x \in [a,b]$ ,  $j = 1,...,m$  . Indeed it is

x 32 The basic background here is as in Remark 20. Also 
$$
{}_{q}M_{j-}(y)
$$
,  $1 \leq q \leq \infty$ ,  $y \in [a,b]$  is as in (80),  
\n
$$
{}^{'}k_{j}^{-}(x,y)
$$
 is as in (83) and  ${}^{P}L_{jq}(x)$  as in (84), where  $x, y \in [a,b]$ . Here it is  
\n
$$
{}^{P}K_{j}^{-}(x,y) := K_{j}(x) = (\psi(b) - \psi(x))^{\xi_{j} - \mu_{j}} E_{\rho_{j},\xi_{j} - \mu_{j}+1}^{-\nu_{j} \omega_{j}} [\omega_{j}(\psi(b) - \psi(x))^{\rho_{j}}]
$$
\n(107)  
\n1,  $j = 1,...,m$ . Indeed it is  
\n
$$
\frac{{}^{P}k_{j}^{-}(x,y)}{{}^{P}K_{j}^{-}(x)} = \left(\chi_{x,b}(y)\psi'(y)\frac{(\psi(y) - \psi(x))^{\xi_{j} - \mu_{j} - 1}}{(\psi(b) - \psi(x))^{\xi_{j} - \mu_{j}}}\right) \left(\frac{E_{\rho_{j},\xi_{j} - \mu_{j}}^{-\nu_{j} \omega_{j}} [\omega_{j}(\psi(y) - \psi(x))^{\rho_{j}}]}{E_{\rho_{j},\xi_{j} - \mu_{j}+1}^{-\nu_{j} \omega_{j}} [\omega_{j}(\psi(b) - \psi(x))^{\rho_{j}}]}\right),
$$
\n(108)  
\n1, b],  $j = 1,...,m$   
\n
$$
\lim_{q} \frac{{}^{P}W_{j-}}{\omega_{j}} \text{ on } [a,b], \text{ with appropriate choice of weight function } u,
$$
\n
$$
\lim_{q} \frac{{}^{P}W_{j-}}{W_{j-}}(y):=_{q} M_{j-}(y)\left(\int_{a}^{y} \frac{u(x)^{p}k_{j}^{-}(x,y)}{{}^{P}K_{j}^{-}(x)}dx\right) < \infty,
$$
\n(109)  
\n1, and that  $\lim_{q} W_{j-}$  is integrable on  $[a,b]$ ;  $j = 1,...,m$ .  
\n20.

 $\forall x, y \in [a, b], j = 1,...,m$ 

We define  ${^{P}_{q}}W_{j-1}$ 

$$
\binom{P}{q}W_{j-}(y):=\n\binom{P}{q}M_{j-}(y)\left(\int_a^y \frac{u(x)^p k_j^-(x, y)}{P K_j^-(x)} dx\right) < \infty,\n\tag{109}
$$

 $\forall \; y \in \left[ a,b \right]$ , and that  $\frac{P}{q} W_{j-}$  is integrable on  $\left[ a,b \right]$ ;  $j=1,...,m.$ 

A direct application of Theorem 25, see also (16), follows:

**Theorem 33** It is all as in Remark 32. Here  $\Phi_j$  : $\mathsf{R}_+^n$   $\rightarrow$   $\mathsf{R}_+$  $j_{ij}$  : $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}^n_+$  ,  $\;j$  =  $1,...,m$  , are convex functions increasing per coordinate. Then

$$
[a,b], j = 1,...,m
$$
\n
$$
\int_{a}^{b} [a,b], \text{ with appropriate choice of weight function } u,
$$
\n
$$
\int_{a}^{b} W_{j-} \text{ on } [a,b], \text{ with appropriate choice of weight function } u,
$$
\n
$$
\int_{a}^{b} W_{j-}(y) :=_{q} M_{j-}(y) \left( \int_{a}^{y} \frac{u(x)^{p} k_{j}^{-}(x, y)}{P_{j-}(x)} dx \right) < \infty,
$$
\n
$$
[109]
$$
\n
$$
\int_{a}^{b} \text{ and that } \int_{a}^{b} W_{j-} \text{ is integrable on } [a,b]; j = 1,...,m.
$$
\n
$$
\int_{a}^{b} \text{ and that } \int_{a}^{b} W_{j-} \text{ is integrable on } [a,b]; j = 1,...,m.
$$
\n
$$
\int_{a}^{b} \text{ and } \int_{a}
$$

### We make

Total Prabhakar Hardy Type Generalized Fractional Inequalities

\nWe make

\n**Remark 34** Let 
$$
f_i \in C([a, b]), i = 1, ..., n, \text{ and } \vec{f} = (f_1, ..., f_n)
$$
. We set

\n
$$
\left\| \vec{f}(y) \right\|_{\infty} := \max \{ |f_1(y)|, ..., |f_n(y)| \}
$$
\nand

\n
$$
\left\| \vec{f}(y) \right\|_{q} := \left( \sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, q \ge 1; y \in [a, b]
$$
\nClearly it is

\n
$$
\left\| \vec{f}(y) \right\|_{q} \in C([a, b]), \text{ for all } 1 \le q \le \infty. \text{ We assume that } \left\| \vec{f}(y) \right\|_{q} > 0, \text{ a.e. on } (a, b), \text{ for } q \in [1, \infty]
$$
\nng fixed.

\nLet

\n
$$
L_q^*(x) := \int_a^x k^*(x, y) \left\| \vec{f}(y) \right\|_{q} dy, x \in [a, b]
$$
\n(112)

\nq ≤ ∞ fixed.

 $q\in C([a,b])$ , for all  $1\leq q\leq\infty$  . We assume that  $\left\|f\big(y\right)\right\|_q\geq 0$  , a.e. on  $(a,b)$ , for  $q\in [1,\infty]$ being fixed.

Let

$$
L_q^+(x) := \int_a^x k^+(x, y) \left\| \overrightarrow{f}(y) \right\|_q dy, \, x \in [a, b], \tag{112}
$$

 $1 \leq q \leq \infty$  fixed.

We assume  $L_q^+(x)\!>\!0$  a.e. on  $(a,b).$ 

Here we considered

$$
L_q^*(x) := \int_a^x k^*(x, y) \left\| \vec{f}(y) \right\|_q dy, x \in [a, b].
$$
\n(d.12)  
\nd.2  
\n
$$
L_q^*(x) > 0 \text{ a.e. on } (a, b).
$$
\n
$$
L_q^*(x) = 0 \text{ a.e. on } (a, b).
$$
\n
$$
k^*(x, y) := k(x, y) := \begin{cases} \n\psi'(y)(\psi(x) - \psi(y))^{\mu-1} E_{\rho, \mu}^* \left[ \omega(\psi(x) - \psi(y))^{\rho} \right] a < y \le x, \\ \n0, x < y < b, \n\end{cases}
$$
\n
$$
Q = 0; \ \psi \in C^1([a, b]) \text{ which is increasing.}
$$
\n
$$
L_q^*(y) := \left\| \vec{f}(y) \right\|_q \left( \int_y^b \frac{u(x)k^*(x, y)}{L_q^*(x)} dx \right) < \infty,
$$
\n
$$
L_q^*(y) = \left\| \vec{f}(y) \right\|_q \left( \int_y^b \frac{u(x)k^*(x, y)}{L_q^*(x)} dx \right) < \infty,
$$
\n
$$
L_q^*(y) = 0 \text{ and that } W_q^*(y) := \left\| \vec{f}(y) \right\|_q \left( \int_y^b \frac{u(x)k^*(x, y)}{L_q^*(x)} dx \right) < \infty,
$$
\n
$$
L_q^*(y) = 0 \text{ and that } W_q^*(y) = 0 \text{ and } W_q^*(y) = 0 \text{ and
$$

where  $\,\rho,\mu,\gamma,\omega\,{>}\,0;\,\psi\in C^1(\![a,b]\!)$  which is increasing.

The weight function  $u$  is chosen so that

$$
W_q^+(y) := \left\| \overrightarrow{f}(y) \right\|_q \left( \int_y^b \frac{u(x)k^+(x, y)}{L_q^+(x)} dx \right) < \infty,
$$
\n(114)

a.e. on  $(a,b)$  and that  $W_q^+$  is integrable on  $\big[a,b\big].$ 

A direct application of Theorem 8 produces:

**Theorem 35** Let all as in Remark 34. Here  $\Phi: R^n_+ \to R$  is a convex and increasing per coordinate function. Then

Journal of Advances in Applied & Computational Mathematics, 8, 2021\n
$$
\int_{a}^{b} u(x) \Phi\left(\frac{\overline{e_{\rho,\mu,\omega,a+}^{z,y'}}(x)}{L_q^+(x)}\right) dx \le \int_{a}^{b} W_q^+(y) \Phi\left(\frac{\overline{f}(y)}{\overline{f}(y)}\right)_q dy.
$$
\n(115)

We make

$$
\int_{a}^{b} u(x) \Phi\left(\frac{\left|e^{\gamma \cdot \psi_{\alpha, \alpha, \alpha}} f(x)\right|}{L_{q}^{\prime}(x)}\right) dx \leq \int_{a}^{b} W_{q}^{+}(y) \Phi\left(\frac{\left|f(y)\right|}{\left|\overline{f}(y)\right|_{q}}\right) dy.
$$
\n
$$
\text{We make}
$$
\n
$$
\text{Remark 36 Let } f_{i} \in C([a, b]), \ i = 1, ..., n, \text{ and } \overline{f} = (f_{1}, ..., f_{n}). \text{ We set}
$$
\n
$$
\left\|\overline{f}(y)\right\|_{\infty} := \max\{f_{i}(y), ..., |f_{n}(y)|\},
$$
\nand\n
$$
\left\|\overline{f}(y)\right\|_{\infty} := \left(\sum_{i=1}^{n} |f_{i}(y)|^{q}\right)^{\frac{1}{q}}, q \geq 1; y \in [a, b]
$$
\n
$$
\text{Clearly it is } \left\|\overline{f}(y)\right\|_{q} \in C([a, b]), \text{ for all } 1 \leq q \leq \infty. \text{ We assume that } \left\|\overline{f}(y)\right\|_{q} > 0, \text{ a.e. on } (a, b), \text{ for } q \in [1, \infty]
$$
\n
$$
L_{q}^{c}(x) := \int_{x}^{b} k^{-}(x, y) \left\|\overline{f}(y)\right\|_{q} dy, x \in [a, b],
$$
\n
$$
q \leq \infty \text{ fixed.}
$$
\n
$$
\text{Let}
$$
\n
$$
L_{q}^{c}(x) := \int_{x}^{b} k^{-}(x, y) \left\|\overline{f}(y)\right\|_{q} dy, x \in [a, b],
$$
\n
$$
q \leq \infty \text{ fixed.}
$$
\n
$$
(117)
$$

 $\mathcal{C}_q\in C([a,b])$ , for all  $1\leq q\leq\infty$  . We assume that  $\left\|f\big(y\big)\right\|_q>0$  , a.e. on  $(a,b)$ , for  $q\in [1,\infty]$ being fixed.

Let

$$
L_q^-(x) := \int_x^b k^-(x, y) \left\| \vec{f}(y) \right\|_q dy, \, x \in [a, b], \tag{117}
$$

 $1 \leq q \leq \infty$  fixed.

We assume  $L_q^-(x)\!>\!0$  a.e. on  $(a,b).$ 

Here we considered

$$
L_q(x) := \int_x^b k^-(x, y) \left\| \vec{f}(y) \right\|_q dy, x \in [a, b],
$$
\nend.

\nbe

\nend.

\nend.

\nend:

\ndefined

\n
$$
k^-(x, y) := k(x, y) := \begin{cases} \psi'(y)(\psi(y) - \psi(x))^{\mu-1} E_{\rho, \mu}^{\gamma} \left[ \omega(\psi(y) - \psi(x))^{\rho} \right] x \leq y < b, \\ 0, a < y < x, \end{cases}
$$
\n(118)

\nfunction  $u$  is chosen so that

\n
$$
W_q^-(y) := \left\| \vec{f}(y) \right\|_q \left( \int_a^y \frac{u(x)k^-(x, y)}{L_q(x)} dx \right) < \infty,
$$
\nand that  $W_q^-$  is integrable on  $[a, b]$ .

\nand the function of Theorem 8 produces:

\n(119)

where  $\,\rho,\mu,\gamma,\omega\,{>}\,0;\,\psi\in C^1(\! [a,b]\!)$  which is increasing.

The weight function  $u$  is chosen so that

$$
W_q^-(y) := \left\| \vec{f}(y) \right\|_q \left( \int_a^y \frac{u(x)k^-(x,y)}{L_q^-(x)} dx \right) < \infty,
$$
\n(119)

a.e. on  $(a,b)$  and that  $\mathit{W}^{-}_{q}$  is integrable on  $[a,b].$ 

A direct application of Theorem 8 produces:

**Theorem 37** Let all as in Remark 36. Here  $\Phi: R^n_+ \to R$  is a convex and increasing per coordinate function. Then

iralized Fractional Inequalities

\nGeorge A. Anastassiou

\n
$$
\int_{a}^{b} u(x) \Phi \left( \frac{\left| \overrightarrow{e}_{p,\mu,\omega,b-}^{y,y} f(x) \right|}{L_{q}^{-}(x)} \right) dx \leq \int_{a}^{b} W_{q}^{-}(y) \Phi \left( \frac{\left| \overrightarrow{f}(y) \right|}{\left| \overrightarrow{f}(y) \right|_{q}} \right) dy.
$$
\ntherical shell:

\n(120)

Next we deal with the spherical shell:

Background 38 We need:

Let  $N\!\geq\! 2$ ,  $S^{N-1}:=\{x\!\in\!{\sf R}^N:|x|\!=\!1\}$  the unit sphere on  $\textsf{\sf R}^N$ , where  $| \cdot |$  stands for the Euclidean norm in  $\textsf{\sf R}^N$ . Also denote the ball  $\,B(0,R)\!:=\!\{x\!\in\!{\sf R}^N\!:\!|x|\!\leq\! R\}\!\subseteq\!{\sf R}^N$  ,  $\,R\!>\!0$  , and the spherical shell

$$
A := B(0, R_2) - \overline{B(0, R_1)}, 0 < R_1 < R_2. \tag{121}
$$

For the following see [12, pp. 149-150], and [13, pp. 87-88].

For 
$$
x \in \mathbb{R}^N - \{0\}
$$
 we can write uniquely  $x = r\omega$ , where  $r = |x| > 0$ , and  $\omega = \frac{x}{r} \in S^{N-1}$ ,  $|\omega| = 1$ .

Clearly here

$$
\mathbf{R}^{N} - \{0\} = (0, \infty) \times S^{N-1}, \tag{122}
$$

and

$$
\overline{A} = [R_1, R_2] \times S^{N-1}.
$$
\n(123)

We will be using

**Theorem 39** ([1, p. 322]) Let  $f : A \rightarrow \mathbb{R}$  be a Lebesgue integrable function. Then

$$
A := B(0, R_2) - \overline{B(0, R_1)}, 0 < R_1 < R_2. \tag{121}
$$
\n9-150], and [13, pp. 87-88].

\nuniquely  $x = r\omega$ , where  $r = |x| > 0$ , and  $\omega = \frac{x}{r} \in S^{N-1}$ ,  $|\omega| = 1$ .

\n
$$
R^N - \{0\} = (0, \infty) \times S^{N-1},
$$
\n
$$
\overline{A} = [R_1, R_2] \times S^{N-1}.
$$
\n
$$
A \rightarrow R \text{ be a Lebesgue integrable function. Then}
$$
\n
$$
\int_A f(x) dx = \int_{S^{N-1}} \left( \int_{R_1}^{R_2} f(r\omega) r^{N-1} dr \right) d\omega.
$$
\n(124)

\nand on the shell in polar form using the polar coordinates  $(r, \omega)$ .

So we are able to write an integral on the shell in polar form using the polar coordinates  $(r,\omega)$ .

We need

**Definition 40** Let  $\rho,\mu,\gamma,w$   $>$   $0;$   $f$   $\in$   $C(\overline{A})$  and  $\psi$   $\in$   $C^1([R_1,R_2])$  which is increasing. The left and right radial Prabhakar fractional integrals with respect to  $\psi$  are defined as follows:

$$
\overline{A} = [R_1, R_2] \times S^{N-1}.
$$
\n(123)  
\np. 322]) Let  $f : A \rightarrow \mathbb{R}$  be a Lebesgue integrable function. Then  
\n
$$
\int_A f(x)dx = \int_{S^{N-1}} (\int_{R_1}^{R_2} f(r\omega)r^{N-1}dr) d\omega.
$$
\n(124)  
\nD write an integral on the shell in polar form using the polar coordinates  $(r, \omega)$ .  
\nLet  $\rho, \mu, \gamma, w > 0; f \in C(\overline{A})$  and  $\psi \in C^1([R_1, R_2])$  which is increasing. The left and right radial  
\nl integrals with respect to  $\psi$  are defined as follows:  
\n
$$
(e_{\rho,\mu,w,R_1}^{rw}, f)(x) = \int_{R_1}^{r} \psi'(t)(\psi(r) - \psi(t))^{\mu-1} E_{\rho,\mu}^{r} [\psi(\psi(r) - \psi(t))^{\rho}] f(t\omega) dt,
$$
\n(125)  
\n
$$
(e_{\rho,\mu,w,R_2}^{rw}, f)(x) = \int_{R_1}^{R_2} \psi'(t)(\psi(t) - \psi(r))^{\mu-1} E_{\rho,\mu}^{r} [\psi(\psi(t) - \psi(r))^{\rho}] f(t\omega) dt,
$$
\n(126)  
\n
$$
S = r\omega, r \in [R_1, R_2], \omega \in S^{N-1}.
$$

and

$$
\left(e_{\rho,\mu,w,R_2}^{y;y} - f(x)\right) = \int_r^{R_2} \psi'(t) (\psi(t) - \psi(r))^{\mu-1} E_{\rho,\mu}^y \left[ w(\psi(t) - \psi(r))^{\rho} \right] f(t\omega) dt, \tag{126}
$$

where  $x\in\overline{A}$  , that is  $\,x=r\omega$  ,  $\,r\in\left[R_1,R_2\right],\;\omega\in S^{N-1}$  .

Based on [1], p. 288 and [2, 4], we have that (125), (126) are continuous functions over  $\overline{A}$  when  $\mu \ge 1$ .

We make

**Remark 41** Let  $f_i \in C[A]$ , where the shell  $\,A\,$  is as in (121),  $\,i=1,...,n$  , and  $\,f=(f_1,...,f_n).$  We set

Journal of Advances in Applied & Computational Mathematics, 8, 2021  
\ne have that (125), (126) are continuous functions over 
$$
\overline{A}
$$
 when  $\mu \ge 1$ .  
\nthe shell  $A$  is as in (121),  $i = 1,...,n$ , and  $\overrightarrow{f} = (f_1,...,f_n)$ . We set  
\n
$$
\left\| \overrightarrow{f}(y) \right\|_{\infty} := \max \left\{ f_1(y), ..., |f_n(y) | \right\},
$$
\nand  
\n
$$
\left\| \overrightarrow{f}(y) \right\|_{q} := \left( \sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, q \ge 1; y \in \overline{A}.
$$
\n $r \le \infty$ . One can write that  
\n
$$
\left\| \overrightarrow{f}(y) \right\|_{q} = \left\| \overrightarrow{f}(t\omega) \right\|_{q}, 1 \le q \le \infty,
$$
\n $v$ , by Background 38.

Clearly it is  $\left\Vert f({\mathbf y})\right\Vert_q \in C[A],\, 1\!\leq\! q\!\leq\!\infty$  . One can write that

$$
\begin{aligned}\n\text{for } \omega \text{ is a constant, } \left\| \vec{f}(y) \right\|_{q} &= \left\| \vec{f}(t\omega) \right\|_{q}, 1 \leq q \leq \infty, \\
\text{by Background 38.} \\
\text{by Background 38.} \\
\text{by } \mathcal{L}(R_{1}, r)(t) \text{ with } t \leq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \chi_{(R_{1}, r)}(t) \text{ with } \forall \left( t \right) \text{ with } t \leq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \chi_{(R_{1}, r)}(t) \text{ with } t \leq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \left( \text{for } R \geq T \text{ and } t \leq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \left( \text{for } R \geq T \text{ and } t \leq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \sum_{k=1}^{R} k_{1}^{+}(r, t) \left\| \vec{f}(t\omega) \right\|_{q} dt, \\
&= \sum_{k=1}^{R} \left( \text{for } R \geq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \sum_{k=1}^{R} k_{1}^{+}(r, t) \left\| \vec{f}(t\omega) \right\|_{q} dt, \\
&= \sum_{k=1}^{R} \left( \text{for } R \geq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \sum_{k=1}^{R} k_{1}^{+}(r, t) \left\| \vec{f}(t\omega) \right\|_{q} dt, \\
&= \sum_{k=1}^{R} \left( \text{for } R \geq T \text{ and } t.\n\end{aligned}
$$
\n
$$
\begin{aligned}\n&= \sum_{k=1}^{R} k_{1}^{+}(r, t) \left\| \vec{f}(t\omega) \right\|_{q} dt, \\
&= \sum_{k=1}^{R} k_{2}^{+}(r, t) \left\| \vec{f
$$

where  $t\in\lbrack R_{\scriptscriptstyle1},R_{\scriptscriptstyle2}\rbrack$ ,  $\omega\!\in\! S^{^{N-1}};\;y\!=\!t\omega$  , by Background 38.

We assume that  $\left\Vert f({\mathcal Y})\right\Vert_q \geq 0$  on  $A$  ,  $1$   $\leq$   $q$   $\leq$   $\infty$  fixed.

Consider the kernel

$$
k_*^+(r,t) := k(r,t) := \chi_{(R_1,r)}(t)\psi'(t)(\psi(r) - \psi(t))^{\mu-1} E_{\rho,\mu}^r \left[ w(\psi(r) - \psi(t))^{\rho} \right]
$$
\n(129)

where  $\,\rho,\mu,\gamma,w\!>\!0;\,\psi\in C^1\big(\!\!\left[R_1,R_2\right]\!\!\right)$  which is increasing.

Let

$$
L_{q*}^+(x) = L_{q*}^+(r\omega) = \int_{R_1}^{R_2} k_*^+(r,t) \left\| \overline{f(t\omega)} \right\|_q dt,
$$
\n(130)

 $x = r\omega \in \overline{A}$ ,  $1 \le q \le \infty$  fixed;  $r \in [R_1, R_2]$ ,  $\omega \in S^{N-1}$ .

We have that  $\, L_{q*}^{\scriptscriptstyle +}(r\omega)\!>\!0\,$  for  $\,r\in (R^{}_1,R^{}_2\,],\,$  fi  $f_*(r\omega) > 0$  for  $r \in (R_1, R_2]$ , for every  $\omega \in S^{N-1}$ .

Here we choose the weight  $u(x) \!=\! u(r\omega) \!=\! L^{+}_{q*}(r\omega).$ 

Consider the function

$$
k(r,t) := \chi_{(R_1,r]}(t)\psi'(t)(\psi(r)-\psi(t))^{\mu-1}E_{\rho,\mu}^{\gamma}\big[\psi(\psi(r)-\psi(t))^{\rho}\big]
$$
\n
$$
[R_1, R_2]\big) \text{ which is increasing.}
$$
\n
$$
L_{q*}^+(x) = L_{q*}^+(r\omega) = \int_{R_1}^{R_2} k_*^+(r,t) \|\widehat{f(t\omega)}\| dt,
$$
\n
$$
\cdot \in [R_1, R_2], \omega \in S^{N-1}.
$$
\n
$$
\text{for } r \in (R_1, R_2], \text{ for every } \omega \in S^{N-1}.
$$
\n
$$
u(x) = u(r\omega) = L_{q*}^+(r\omega).
$$
\n
$$
W_{q*}^+(y) = W_{q*}^+(t\omega) = \left\|\widehat{f(t\omega)}\right\|_q \left(\int_{R_1}^{R_2} k_*^+(r,t)dr\right) < \infty,
$$
\n
$$
W_{q*}^+(t\omega) \text{ is integrable over } [R_1, R_2], \forall \omega \in S^{N-1}.
$$
\n
$$
\text{for each } r \in (R_1, R_2].
$$
\n
$$
(131)
$$

 $\forall$   $t\in\left[R_1,R_2\right]$ ,  $\omega\in S^{\scriptscriptstyle N-1};$  and  $W_{q*}^+(t\omega)$  is integrable over  $\big[R_1,$  $\Gamma_{*}^{+}(t\omega)$  is integrable over  $\left[ R_1,R_2\right]$ ,  $\forall\;\;\omega \in S^{N-1}.$ 

Here  $\Phi$  :  $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}$  is a convex and increasing per coordinate function. By (115) we obtain

pe Generalized Fractional Inequalities  
\n
$$
\int_{R_1}^{R_2} L_{q*}^+(r\omega) \Phi\left(\frac{\left|\overline{e_{\rho,\mu,w,R_1+}^{r,w}f(r\omega)}\right|}{L_{q*}^+(r\omega)}\right) dr \le \int_{R_1}^{R_2} W_{q*}^+(t\omega) \Phi\left(\frac{\left|\overline{f(t\omega)}\right|}{\left|\overline{f(t\omega)}\right|_q}\right) dt, \tag{132}
$$
\n
$$
\le r \le R \quad \text{and} \quad R^{N-1} \le r^{N-1} \le R^{N-1} \quad \text{and} \quad R^{1-N} \le r^{1-N} \le R^{1-N} \quad \text{also} \quad r^{N-1}r^{1-N} = 1. \text{ Thus by (132)}
$$

 $\forall \omega \in S^{N-1}.$ 

Here we have  $R_1 \le r \le R_2$ , and  $R_1^{N-1} \le r^{N-1} \le R_2^{N-1}$  $1 \times N-1$  $R_1^{N-1} \le r^{N-1} \le R_2^{N-1}$ , and  $R_2^{1-N} \le r^{1-N} \le R_1^{1-N}$ 1  $1 - N$   $\sim$   $\frac{1}{2}$  $2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$ , also  $r^{N-1}r^{1-N} = 1$ . Thus by (132), we have

$$
\int_{R_1}^{R_2} L_{q*}^+(r\omega) \Phi\left(\frac{\left|\frac{e^{\gamma \cdot w}}{e^{\gamma \cdot w \cdot R_1 + f(r\omega)}}\right|}{L_{q*}^+(r\omega)}\right) dr \leq \int_{R_1}^{R_2} W_{q*}^+(t\omega) \Phi\left(\frac{\left|\frac{f(t\omega)}{f(t\omega)}\right|}{\left|\frac{f(t\omega)}{f(t\omega)}\right|_{q}}\right) dt, \qquad (132)
$$
\nhave  $R_1 \leq r \leq R_2$ , and  $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$ , and  $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$ , also  $r^{N-1}r^{1-N} = 1$ . Thus by (132),

\n
$$
\int_{R_1}^{R_2} L_{q*}^+(r\omega) \Phi\left(\frac{\left|\frac{e^{\gamma \cdot w}}{e^{\gamma \cdot w \cdot R_1 + f(r\omega)}}\right|}{L_{q*}^+(r\omega)}\right) r^{N-1} dr \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_{R_1}^{R_2} W_{q*}^+(r\omega) \Phi\left(\frac{\left|\frac{f(r\omega)}{f(r\omega)}\right|}{\left|\frac{f(r\omega)}{f(r\omega)}\right|_{q}}\right) r^{N-1} dr, \qquad (133)
$$
\nit holds

\nif holds

 $\forall \omega \in S^{N-1}.$ 

Therefore it holds

$$
\int_{a_1}^{a_2} L_{q*}^*(r\omega) \Phi\left(\frac{e_{\rho,w,n,\beta_1}^{r,w} f(r\omega)}{L_{q*}^*(r\omega)}\right) dr \leq \int_{a_1}^{a_2} W_{q*}^*(t\omega) \Phi\left(\frac{\sqrt{f(\omega)}}{\sqrt{f(t\omega)}}\right) dt,
$$
\n
$$
\forall \omega \in S^{N-1}.
$$
\nHere we have  $R_i \leq r \leq R_2$ , and  $R_i^{N-1} \leq r^{N-1} \leq R_i^{N-1}$ , and  $R_i^{1-N} \leq r^{1-N} \leq R_i^{1-N}$ , also  $r^{N-1}r^{1-N} = 1$ . Thus by (132), we have\n
$$
\int_{a_1}^{a_2} L_{q*}^*(r\omega) \Phi\left(\frac{\left|e_{\rho,w,n,\beta_1}^{r,w} f(r\omega)\right|}{L_{q*}^*(r\omega)}\right) r^{N-1} dr \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_{a_1}^{a_2} W_{q*}^*(r\omega) \Phi\left(\frac{\sqrt{f(r\omega)}}{\sqrt{f(r\omega)}}\right) r^{N-1} dr,
$$
\nTherefore it holds\n
$$
\int_{S^{N-1}} \left(\int_{a_1}^{a_2} L_{q*}^*(r\omega) \Phi\left(\frac{\left|e_{\rho,w,n,\beta_1}^{r,w} f(r\omega)\right|}{L_{q*}^*(r\omega)}\right) r^{N-1} dr\right) d\omega \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_{S^{N-1}} \left(\int_{a_1}^{a_2} W_{q*}^*(r\omega) \Phi\left(\frac{\sqrt{f(r\omega)}}{\sqrt{f(r\omega)}}\right) r^{N-1} dr\right) d\omega.
$$
\n
$$
\text{Using Theorem 33 we derive:}
$$
\n
$$
\text{Theorem 42 All as in Remark 41. Then}
$$
\n
$$
\int_{s^{N-1}} \left(\frac{e_{\rho,w,n,\beta_1}^{r,w} f(x)}{L_{q*}^*(x)} \right) \left(\frac{e_{\rho,w,n,\beta_1}^{r,w} f(x)}{L_{q*}^*(x)}\right) dx \leq \left(\frac{R_2
$$

Using Theorem 39 we derive:

Theorem 42 All as in Remark 41. Then

$$
\int_{A} L_{q*}^{+}(x) \Phi\left(\frac{\left|\overline{e_{\rho,\mu,w,R_{1}}^{y;y'}},f(x)\right|}{L_{q*}^{+}(x)}\right) dx \leq \left(\frac{R_{2}}{R_{1}}\right)^{N-1} \int_{A} W_{q*}^{+}(x) \Phi\left(\frac{\left|\overline{f(x)}\right|}{\left\|\overline{f(x)}\right\|_{q}}\right) dx, \tag{135}
$$

 $\gamma;\psi$  $\rho,\mu$  $\gamma;\psi$  $\rho,\mu$  $\gamma;\psi$  $\rho,\mu$ ;  $1 \mathcal{N}$ ,  $\mathcal{N}$ ,  $\mathcal{C}_{\rho,\mu,w,R_1}$ ;  $, \mu, w, R_1$ ;  $\mathcal{L}_{1,\mu,w,R_1+}^{i,w}f(x)\!=\! (\!(\!e_{\rho,\mu,w,R_1+}^{y,w},\!f_1\!)\!(x),\!..., \!(\!e_{\rho,\mu,w,R_1+}^{y,w},\!f_n\!)\!(x))$  and coordinates are assumed to be continuous functions on  $\overline{A}$ .

We make

**Remark 43** Let  $f_i \in C[A]$ , where the shell  $\,A\,$  is as in (121),  $\,i=1,...,n$  , and  $\,f=(f_1,...,f_n).$  We set

$$
\int_{\gamma,\mu,\text{w},R_1+} f(x) \text{d}x \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_{A} W_{q*}^{+}(x) \Phi\left(\frac{\overline{f(x)}}{\overline{f(x)}}\right) dx, \qquad (135)
$$
\n
$$
\int_{\gamma,\mu,\text{w},R_1+} \left(e_{\rho,\mu,\text{w},R_1+} \int_{R_1} f(x) \text{d}x\right) dx \qquad (136)
$$
\n
$$
\int_{\gamma,\mu,\text{w},R_1+} \left(e_{\rho,\mu,\text{w},R_1+} \int_{R_1} f(x) \text{d}x\right) dx \qquad \text{for } \mu \text{ and } \mu \text{ is a constant, and } \mathcal{F} = \int_{\gamma,\text{w},R_1} f(x) \text{d}x \qquad (137)
$$
\n
$$
\text{Let } \mathbf{R} \text{ is a linear in } \mathbb{E}[f_1(x)] \text{ and } \mathbb{E}[f_2(x)] \text{ and } \mathbb{E}[f_1(x)] \text{ and } \mathbb{E}[f_2(x)] \text{ and } \mathbb{E}[
$$

Clearly it is  $\left\Vert f({\mathcal Y})\right\Vert_q \in C[A],\, 1\leq q \leq \infty$  . One can write that

\n Journal of Advances in Applied & Computational Mathematics, 8, 2021\n

\n\n
$$
\left\| \vec{f}(y) \right\|_q = \left\| \vec{f}(t\omega) \right\|_q, 1 \leq q \leq \infty,
$$
\n

\n\n W Background 38.\n

where  $t\in\left[ R_{{\scriptscriptstyle 1}},R_{{\scriptscriptstyle 2}} \right]$ ,  $\omega\!\in\! S^{{\scriptscriptstyle N}-1}\!$ ;  $y\!=\!t\omega$  , by Background 38.

We assume that 
$$
\left\|\vec{f}(y)\right\|_q > 0
$$
 on  $\overline{A}$ ,  $1 \le q \le \infty$  fixed.

Consider the kernel

$$
k_*^-(r,t) := k(r,t) := \chi_{[r,R_2)}(t)\psi'(t)(\psi(t) - \psi(r))^{\mu-1} E_{\rho,\mu}^{\gamma}\big[\psi(\psi(t) - \psi(r))^{\rho}\big]
$$
(138)

where  $\,\rho,\mu,\gamma,w\!>\!0;\,\psi\in C^1\big(\!\!\left[R_1,R_2\right]\!\!\right)$  which is increasing.

Let

Journal of Advances in Applied & Computational Mathematics, 8, 2021\n
$$
\left\| \vec{f}(y) \right\|_q = \left\| \vec{f}(t\omega) \right\|_q, 1 \le q \le \infty,
$$
\n(137)\n
$$
d\omega, \text{ by Background 38.}
$$
\n
$$
\vec{A}, 1 \le q \le \infty \text{ fixed.}
$$
\n
$$
\vec{A}, 1 \le q \le \infty \text{ fixed.}
$$
\n
$$
\sum_{i=1}^{n} \sum_{r=1}^{n} (t\omega)^{i} (t) \psi(t) (\psi(t) - \psi(r))^{i-1} E_{\rho,\mu}^{r} \left[ w(\psi(t) - \psi(r))^{i} \right].
$$
\n(138)\n
$$
R_1, R_2] \text{ which is increasing.}
$$
\n
$$
L_{q*}(x) = L_{q*}(r\omega) = \int_{R_1}^{R_2} k_*^{-}(r, t) \left\| \vec{f}(t\omega) \right\|_q dt,
$$
\n
$$
R_1, R_2] \text{, } \omega \in S^{N-1}.
$$
\n(139)

 $x = r\omega \in \overline{A}$ ,  $1 \le q \le \infty$  fixed;  $r \in [R_1, R_2]$ ,  $\omega \in S^{N-1}$ .

We have that  $\, L_{q*}^-(r\omega)\!>\!0\,$  for  $\,r\in (R^{}_1,R^{}_2\,],\,$  fi  $f_*(r\omega) > 0$  for  $r \in (R_1, R_2]$ , for every  $\omega \in S^{N-1}$ .

Here we choose the weight  $u(x)$  =  $u(r\omega)$  =  $L_{q*}^-(r\omega)$ .

Consider the function

$$
k(r,t) := \chi_{[r,R_2)}(t)\psi'(t)(\psi(t) - \psi(r))^{\alpha-1} E_{\rho,\mu}^{\gamma} \left[ w(\psi(t) - \psi(r))^{\rho} \right]
$$
\n
$$
R_1, R_2] \text{ which is increasing.}
$$
\n
$$
L_{q*}(x) = L_{q*}(r\omega) = \int_{R_1}^{R_2} k_*^-(r,t) \left\| \overline{f(t\omega)} \right\| dt,
$$
\n
$$
\cdot \in [R_1, R_2], \omega \in S^{N-1}.
$$
\n
$$
\text{for } r \in (R_1, R_2], \text{ for every } \omega \in S^{N-1}.
$$
\n
$$
u(x) = u(r\omega) = L_{q*}(r\omega).
$$
\n
$$
W_{q*}(y) = W_{q*}(t\omega) = \left\| \overline{f(t\omega)} \right\|_q \left( \int_{R_1}^{R_2} k_*^-(r,t) dr \right) < \infty,
$$
\n
$$
W_{q*}(t\omega) \text{ is integrable over } [R_1, R_2], \forall \omega \in S^{N-1}.
$$
\n
$$
\text{for each } r \in (R_1, R_2].
$$
\n
$$
\text{(140)}
$$

 $\forall$   $t\in\left[R_1,R_2\right]$ ,  $\omega\in S^{\scriptscriptstyle N-1};$  and  $W_{q*}^-(t\omega)$  is integrable over  $\big[R_1,$  $I_{*}^{-}(t\omega)$  is integrable over  $\left[ R_{{\rm l}},R_{{\rm 2}}\right]$ ,  $\ \forall \ \ \omega \,{\in}\, S^{{\rm N}-1}.$ 

Here  $\Phi: R_{+}^{n} \rightarrow R$  is a convex and increasing per coordinate function. By (120) we obtain

$$
L_{q*}(x) = L_{q*}(r\omega) = \int_{R_1}^{R_2} k_*^-(r,t) \left\| \overline{f(t\omega)} \right\|_q dt,
$$
\n
$$
\text{fixed: } r \in [R_1, R_2], \omega \in S^{N-1}.
$$
\n
$$
\omega) > 0 \text{ for } r \in (R_1, R_2], \text{ for every } \omega \in S^{N-1}.
$$
\n
$$
\text{weight } u(x) = u(r\omega) = L_{q*}(r\omega).
$$
\n
$$
W_{q*}(y) = W_{q*}(t\omega) = \left\| \overline{f(t\omega)} \right\|_q \left( \int_{R_1}^{R_2} k_*^-(r,t) dr \right) < \infty,
$$
\n
$$
W_{q*}(y) = W_{q*}(t\omega) \text{ is integrable over } [R_1, R_2], \forall \omega \in S^{N-1}.
$$
\n
$$
\text{s a convex and increasing per coordinate function. By (120) we obtain}
$$
\n
$$
\int_{R_1}^{R_2} L_{q*}(r\omega) \Phi \left( \frac{\left| \overline{f_{r,\mu_1,\nu_1,\mu_2}} - f(r\omega) \right|}{L_{q*}(r\omega)} \right) dr \le \int_{R_1}^{R_2} W_{q*}(t\omega) \Phi \left( \frac{\left| \overline{f(t\omega)} \right|}{\left| \overline{f(t\omega)} \right|_q} \right) dt,
$$
\n
$$
r \le R_2, \text{ and } R_1^{N-1} \le r^{N-1} \le R_2^{N-1}, \text{ and } R_2^{1-N} \le r^{1-N} \le R_1^{1-N}, \text{ also } r^{N-1}r^{1-N} = 1. \text{ Thus by (141),}
$$
\n
$$
\text{A}\Phi \left( \frac{\left| \overline{f_{r,\mu_1,\nu_1,\mu_2}} - f(r\omega) \right|}{L_{q*}(r\omega)} \right) \left| r^{N-1} dr \le \left( \frac{R_2}{R_1} \right)^{N-1} \int_{R_1}^{R_2} W_{q*}^-(r\omega) \Phi \left( \frac{\left| \overline{f(r\omega)} \right|}{\left| \overline{f(r\omega)} \right|_q} \right) r
$$

 $\forall \omega \in S^{N-1}.$ 

Here we have  $R_1 \le r \le R_2$ , and  $R_1^{N-1} \le r^{N-1} \le R_2^{N-1}$  $1 \times N-1$  $R_1^{N-1} \le r^{N-1} \le R_2^{N-1}$ , and  $R_2^{1-N} \le r^{1-N} \le R_1^{1-N}$ 1  $1 - N$   $\sim$   $\frac{1}{2}$  $2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$ , also  $r^{N-1}r^{1-N} = 1$ . Thus by (141), we have

$$
W_{q*}^{-}(y) = W_{q*}^{-}(t\omega) = \left\| \overline{f(t\omega)} \right\|_{q} \left( \int_{R_{1}}^{R_{2}} k_{*}^{-}(r, t) dr \right) < \infty, \qquad (140)
$$
\n
$$
P_{2}^{-}(t\omega) = \left\| \overline{f(t\omega)} \right\|_{q} \left( \int_{R_{1}}^{R_{2}} k_{*}^{-}(r, t) dr \right) < \infty, \qquad (140)
$$
\n
$$
P_{1}^{n} \rightarrow \mathbb{R} \text{ is a convex and increasing per coordinate function. By (120) we obtain}
$$
\n
$$
\int_{R_{1}}^{R_{2}} L_{q*}^{-}(r\omega) \Phi \left( \frac{\overline{e_{\rho,\mu,\mathbf{w},R_{2}}^{r}} f(r\omega)}{\overline{L_{q*}}(r\omega)} \right) dr \leq \int_{R_{1}}^{R_{2}} W_{q*}^{-}(t\omega) \Phi \left( \frac{\overline{f(t\omega)}}{\overline{f(t\omega)}} \right) dt, \qquad (141)
$$
\n
$$
P_{2}^{-}(t\omega) \Phi \left( \frac{\overline{e_{\rho,\mu,\mathbf{w},R_{2}}^{r}} f(r\omega)}{\overline{L_{q*}}(r\omega)} \right) dr \leq \int_{R_{1}}^{R_{2}} W_{q*}^{-}(t\omega) \Phi \left( \frac{\overline{f(t\omega)}}{\overline{f(t\omega)}} \right) dt, \qquad (141)
$$
\n
$$
P_{3}^{-1} \leq r \leq R_{2}, \text{ and } R_{1}^{-1} \leq r^{N-1} \leq R_{2}^{N-1}, \text{ and } R_{2}^{1-N} \leq r^{1-N} \leq R_{1}^{1-N}, \text{ also } r^{N-1}r^{1-N} = 1. \text{ Thus by (141)},
$$
\n
$$
\int_{R_{1}}^{R_{2}} L_{q*}^{-}(r\omega) \Phi \left( \frac{\overline{e_{\rho,\mu,\mathbf{w},R_{2}}^{r}} f(r\omega)}{\overline{L_{q*}}(r\omega)} \right) r^{N-1} dr \leq \left( \frac{R_{2}}{R_{1}} \right)^{N-1} \int_{R_{1}}^{R_{2}} W_{q*}^{-}(r\omega) \Phi
$$

 $\forall \omega \in S^{N-1}.$ 

Therefore it holds

Vectorial Prabhakar Hardy Type Generalized Fractional Inequalities\n\nTherefore it holds\n
$$
\int_{S^{N-1}} \left( \int_{R_1}^{R_2} L_{q*}(r\omega) \Phi \left( \frac{\left| \frac{e^{\gamma, W_{\nu, w, R_2}}}{L_{q*}(r\omega)} \int r^{N-1} dr \right| d\omega \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{S^{N-1}} \left( \int_{R_1}^{R_2} W_{q*}^{-}(r\omega) \Phi \left( \frac{\left| \overline{f(r\omega)} \right|}{\left| \overline{f(r\omega)} \right|_{q}} \right| r^{N-1} dr \right) d\omega. \quad (143)
$$
\nUsing Theorem 39 we derive:\n\n**Theorem 44** All as in Remark 43. Then\n\n
$$
\int_{A} L_{q*}(x) \Phi \left( \frac{\left| \frac{e^{\gamma, W_{\nu, w, R_2}}}{L_{q*}(x)} \right|}{L_{q*}(x)} \right) dx \leq \left( \frac{R_2}{R_1} \right)^{N-1} \int_{A} W_{q*}^{-}(x) \Phi \left( \frac{\left| \overline{f(r\omega)} \right|}{\left| \overline{f(x)} \right|_{q}} \right) dx, \quad (144)
$$
\n\nwhere\n
$$
\frac{e^{\gamma, W_{\nu, w, R_2}}}{e^{\gamma, W_{\nu, R_2}} \int_{A} x} \int_{B} = \left( e^{\gamma, W_{\nu, w, R_2}} \int_{A} x \right) \int_{A} ... , \left( e^{\gamma, W_{\nu, w, R_2}} \int_{A} x \right) dx.
$$
\n\nWe need\n\nWe need

Using Theorem 39 we derive:

Theorem 44 All as in Remark 43. Then

$$
\int_A L_{q*}^{-}(x) \Phi\left(\frac{\left|\overline{e_{\rho,\mu,w,R_2}^{y;\psi}-f(x)}\right|}{L_{q*}^{-}(x)}\right) dx \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_A W_{q*}^{-}(x) \Phi\left(\frac{\left|\overline{f(x)}\right|}{\left\|\overline{f(x)}\right\|_q}\right) dx, \tag{144}
$$

 $\gamma;\psi$  $\rho,\mu$  $\gamma;\psi$  $\rho,\mu$  $\gamma;\psi$  $\rho,\mu$ ;  $1 \mathcal{N}$   $\mathcal{N}$   $\cdots$ ,  $\mathfrak{e}_{\rho,\mu,w,R_2}$ ;  $, \mu, w, R_2$ ;  $\mathcal{L}_{\mu,\mu,\kappa,R_2-}^{\mu\nu}f(x)\!=\! \big([\!e_{\rho,\mu,\kappa,R_2-}^{\gamma;\nu}\!f_1\!\big](\!x),\!..., \!\big(\!e_{\rho,\mu,\kappa,R_2-}^{\gamma;\nu}\!f_n\!\big)(\!x)\big)$  and coordinates are assumed to be continuous functions on  $A$ .

We need

**Definition 45** Let  $\rho,\mu,w \geq 0, \ \gamma < 0$ ,  $N = \lceil \mu \rceil$ ,  $\mu \notin \mathsf{N};\ f \in C^N\big(\overline{A}\big)$  and  $\psi \in C^N\big(\llbracket R_1,R_2 \rrbracket\big),\psi^{'}(r) \neq 0$ ,  $\forall$  $r\in[R_1,R_2]$ , and  $\psi$  is increasing. We define the  $\psi$  -Prabhakar-Caputo radial left and right fractional derivatives of order  $\mu$  as follows (  $x\in\overline{A}$  ;  $\,x=r\varpi$  ,  $\,r\in\left[R_1,R_2\right],\,\varpi\in S^{N-1})$ 

as in Remark 43. Then  
\n
$$
\int_{A} L_{q*}(x) \Phi\left(\frac{\left|e_{\rho,u,w,R_{2}}^{xy} - f(x)\right|}{L_{q*}(x)}\right) dx \leq \left(\frac{R_{2}}{R_{1}}\right)^{N-1} \int_{A} W_{q*}(x) \Phi\left(\frac{\left|f(x)\right|}{\left|f(x)\right|_{q}}\right) dx,
$$
\n(144)  
\n
$$
\overline{x}) = \left(\left(e_{\rho,u,w,R_{2}}^{xy} - f_{1}\right)x,...,\left(e_{\rho,u,w,R_{2}}^{xy} - f_{n}\right)x\right) \text{ and coordinates are assumed to be continuous}
$$
\nLet  $\rho, \mu, w > 0, \gamma < 0, N = \left[\mu\right], \mu \notin \mathbb{N}, f \in C^{N}\left(\overline{A}\right) \text{ and } \psi \in C^{N}\left(\left[R_{1},R_{2}\right], \psi\left(r\right) \neq 0, \forall \overline{A} \text{ is increasing. We define the } \Psi \text{-Prabhakar-Gaputo radial left and right fractional derivatives of}$ \n
$$
(x \in \overline{A}; x = r\omega, r \in \left[R_{1},R_{2}\right], \omega \in S^{N-1})
$$
\n
$$
\left(\frac{c}{\kappa}D_{\rho,u,w,R_{1}+}^{xy}f\right)x = \left(\frac{c}{\kappa}D_{\rho,u,w,R_{1}+}^{xy}f\right)r\omega\right):=
$$
\n
$$
\int_{R_{1}}^{r}\psi'(t)(\psi(r) - \psi(t))^{N-\mu-1}E_{\rho,N-\mu}^{xy}[\psi(\psi(r) - \psi(t))^{\nu}\left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^{N}f(t\omega)dt
$$
\n(145)  
\n
$$
\left(\frac{125}{\epsilon}e_{\rho,N-\mu,w,R_{1}+}^{xy}f_{\psi}^{[N]}\right)x,
$$
\n
$$
f_{\psi}^{[N]}(x) = f_{\psi}^{[N]}(r\omega):= \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^{N}f(r\omega),
$$
\n
$$
f_{\psi}^{[N]}(x) = f_{\psi}^{[N]}(r\omega):= \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^{N}f(r\omega
$$

where

$$
f_{\psi}^{[N]}(x) = f_{\psi}^{[N]}(r\omega) := \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^{N} f(r\omega),\tag{146}
$$

is the N th order  $\psi$  -radial derivative of  $f$ ,

and

$$
\left(\,_{R}^{C}D_{\rho,\mu,w,R_{1}+}^{r,\nu}f\right)\!x\right) = \left(\,_{R}^{C}D_{\rho,\mu,w,R_{1}+}^{r,\nu}f\right)\!/r\omega\right) :=
$$
\n
$$
\left(\,_{W}(r) - \psi\left(t\right)\right)^{N-\mu-1}E_{\rho,N-\mu}^{-\nu}\left[\psi\left(\psi\left(r\right) - \psi\left(t\right)\right)^{\rho}\left(\frac{1}{\psi^{\prime}\left(r\right)}\frac{d}{dr}\right)^{N}f\left(\tau\omega\right)dt\right]
$$
\n
$$
\stackrel{\text{(125)}}{=} \left(\,e_{\rho,N-\mu,w,R_{1}+}^{-\nu}f_{\psi}^{[N]}\right)\!x\right),
$$
\n
$$
f_{\psi}^{[N]}(x) = f_{\psi}^{[N]}(r\omega) := \left(\frac{1}{\psi^{\prime}\left(r\right)}\frac{d}{dr}\right)^{N}f\left(\tau\omega\right),
$$
\n
$$
\text{derivative of } f,
$$
\n
$$
\left(\,_{R}^{C}D_{\rho,\mu,w,R_{2}-}^{r,\nu}f\right)\!x\right) = \left(\,_{R}^{C}D_{\rho,\mu,w,R_{2}-}^{r,\nu}f\right)\!r\omega\right) :=
$$
\n
$$
\left(-1\right)^{N}\int_{r}^{R_{2}}\!\psi^{\prime}\left(t\right)\!\left(\psi\left(t\right) - \psi\left(r\right)\right)^{N-\mu-1}E_{\rho,N-\mu}^{-\nu}\left[\psi\left(\psi\left(t\right) - \psi\left(r\right)\right)^{\rho}\right]
$$
\n
$$
\left(\frac{1}{\psi^{\prime}\left(r\right)}\frac{d}{dr}\right)^{N}f\left(\tau\omega\right)dt\right) = (-1)^{N}\left(\,e_{\rho,N-\mu,w,R_{2}-}^{-\nu}f_{\psi}^{[N]}\right)\!x\right),
$$
\n
$$
\mathbf{s}_{2}
$$

 $\forall x \in \overline{A}$ .

In this work we assume that 
$$
\binom{C}{R} D_{\rho,\mu,w,R_1+}^{y;y} f
$$
 and  $\binom{C}{R} D_{\rho,\mu,w,R_2-}^{y;y} f$  are continuous functions over  $\overline{A}$ .

We make

**Remark 46** Let  $\rho,\mu,w > 0, \, \gamma < 0$  ,  $N = \lceil \mu \rceil, \quad \mu \notin \mathsf{N};\, \ f_i \in C^N \! \left( \overline{A} \right) \, i = 1,...,n, \quad \text{and} \quad \overrightarrow{f} = (f_1,...,f_n), \quad \text{and}$  $\psi \in C^N([R_1, R_2])$ ,  $\psi'(r) \neq 0$ ,  $\forall r \in [R_1, R_2]$  and  $\Psi$  is increasing. We follow Definition 45 and we set:

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\n
$$
\int_{C} D_{\rho,\mu,w,R_1+}^{xy} f \, dm \left( \int_{R}^{C} D_{\rho,\mu,w,R_2-}^{xy} f \right) \text{ are continuous functions over } \overline{A}.
$$
\n
$$
D, \gamma < 0, \quad N = \lceil \mu \rceil, \quad \mu \notin \mathbb{N}; \ f_i \in C^N(\overline{A}), i = 1, \dots, n, \quad \text{and } \overrightarrow{f} = (f_1, \dots, f_n), \quad \text{and}
$$
\n
$$
\in [R_1, R_2], \text{ and } \psi \text{ is increasing. We follow Definition 45 and we set:}
$$
\n
$$
\left\| \overline{f_v^{[N]}(y)} \right\|_{\infty} := \max \left\{ f_{1v}^{[N]}(y) \Big|, \dots, \left| f_{nv}^{[N]}(y) \right| \right\}
$$
\nand\n
$$
\left\| \overline{f_v^{[N]}(y)} \right\|_{q} := \left( \sum_{i=1}^{n} \left| f_{iv}^{[N]}(y) \right|^{q} \right)^{\frac{1}{q}}, q \ge 1; y \in \overline{A}.
$$
\n
$$
\left\| \overline{f_v^{[N]}(y)} \right\|_{q} = \left\| \overline{f_v^{[N]}(t\omega)} \right\|_{q}, 1 \le q \le \infty, \tag{149}
$$

One can write that

$$
\left\| \overrightarrow{f_{\psi}^{[N]}(y)} \right\|_{q} = \left\| \overrightarrow{f_{\psi}^{[N]}}(t\omega) \right\|_{q}, 1 \le q \le \infty,
$$
\n(149)

where  $t\in[R_{1},R_{2}]$  ,  $\omega\in S^{N-1}$  ;  $\mathcal{Y}=t\omega$  . We make<br> **Remark 46** Let  $\rho, \mu, w > 0, \gamma < 0$ ,  $N = \lceil \mu \rceil$ ,  $\mu \notin \mathbb{N}$ ;  $f_i \in C^{\infty}(\overline{A})$   $i = 1,...,n$ , and<br>  $\in C^{\infty}([R_1, R_2], \psi'(r) \neq 0, \forall r \in [R_1, R_2],$  and  $\psi$  is increasing. We follow Definition 45 and<br>  $\frac{1}{\sqrt{\frac{1}{\gamma$ q  $C_{\psi}^{[N]}(y)$   $\in C(\overline{A}), 1 \leq q \leq \infty.$  $\mathcal{L} \in C^{\infty}([R_1, R_2], \psi'(r) \neq 0, \forall r \in [R_1, R_2],$  and  $\psi$  is increasing. We follow Definition 45 and we so<br>  $\left\| \frac{f_{\psi}^{(N)}(y)}{f_{\psi}^{(N)}(y)} \right\|_{\mathcal{L}} := \max \left\{ f_{\psi}^{(N)}(y) \right\}, \dots, \left| f_{\psi}^{(N)}(y) \right\} \right\}$ <br>
and<br>  $\left\| \frac{f_{$ q  $\|f_{\psi}^{[N]}(y)\|>0$  on  $\overline{A}$  ,  $1\!\leq\!q\!\leq\!\infty$  fixed.  $\left\| \overline{f_{\psi}^{[N]}(y)} \right\|_{q} = \left\| \overline{f_{\psi}^{[N]}}(t\omega) \right\|_{q}, 1 \leq q \leq \infty,$  (149)<br>  $\omega$ .<br>  $\leq q \leq \infty.$ <br>  $\lambda \in \mathcal{A} \leq \omega \text{ fixed.}$ <br>  $\chi_{(R_{1},r)}(t)\psi'(t)(\psi(r)-\psi(t))^{N-\mu-1} E_{\rho,N-\mu}^{-\gamma} \left[ w(\psi(r)-\psi(t))^{p} \right]$  (150)<br>  $(x) =^{c} L_{q}^{*}(r\omega) = \int$  $\begin{aligned} &\frac{1}{\|v\|_{\mathcal{V}}(v)}\left\|_{q} = \left\|\overline{f_{\psi}^{[N]}}(t\omega)\right\|_{q}, 1 \leq q \leq \infty, \right. \\ &\leq \infty, \\ &\bar{q}, 1 \leq q \leq \infty \text{ fixed.} \\ &\bar{q}, 1 \leq q \leq \infty \text{ fixed.} \end{aligned}$ (149)<br>  $\begin{aligned} &\leq \infty, \\ &\bar{q}, 1 \leq q \leq \infty \text{ fixed.} \end{aligned}$ <br>  $\begin{aligned} &\int_{[1,t]}(t)\psi'(t)(\$ 

Consider the kernel

$$
{}^{C}k^{+}(r,t):=k(r,t):=\chi_{(R_{1},r)}(t)\psi^{'}(t)(\psi(r)-\psi(t))^{N-\mu-1}E_{\rho,N-\mu}^{-\gamma}\bigg[\psi(\psi(r)-\psi(t))^{\rho}\bigg]
$$
\n(150)

Let

$$
{}^{C}L_{q}^{+}(x) = {}^{C}L_{q}^{+}(r\omega) = \int_{R_{1}}^{R_{2}} {}^{C}k^{+}(r,t) \left\| \overline{f_{\psi}^{[N]}(t\omega)} \right\|_{q} dt,
$$
\n(151)

 $x \,{=}\, r \omega \,{\in}\, \overline{A}$  ,  $1 \,{\leq}\, q \,{\leq}\, \infty$  fixed;  $r \,{\in}\, [R_{\rm l},R_{\rm 2}]$  ,  $\omega \,{\in}\, S^{\scriptscriptstyle N-1}.$ 

We have that  ${}^{C}L_{q}^{+}(r\omega)$   $>$   $0$  for  $r$   $\in$   $(R_{1}, R_{2}]$ ,  $\forall$   $\omega$   $\in$   $S^{N-1}.$ 

Here we choose the weight  $u(x) \!= u(r\omega) \!=\!~^C L_q^{\!+}(r\omega).$ 

Consider the function

$$
k(r,t) := \chi_{(R_1,r)}(t)\psi'(t)(\psi(r) - \psi(t))^{N-\mu-1} E_{\rho,N-\mu}^{-r} [\psi(\psi(r) - \psi(t))^{\rho}]
$$
\n
$$
{}^{c}L_{q}^{+}(x) = {}^{c}L_{q}^{+}(r\omega) = \int_{R_1}^{R_2} k^{+}(r,t) \left\| \mathcal{F}_{\psi}^{[N]}(t\omega) \right\|_{q} dt,
$$
\n
$$
t; r \in [R_1, R_2], \omega \in S^{N-1}.
$$
\n
$$
0 \text{ for } r \in (R_1, R_2], \forall \omega \in S^{N-1}.
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \psi'(t\omega)} = \int_{R_1}^{R_2} k^{+}(r,t) \, dr \, dt,
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_1}^{R_2} k^{+}(r,t) \, dr \, dt,
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_1}^{R_2} k^{+}(r,t) \, dr \, dt,
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_1}^{R_2} k^{+}(r,t) \, dr \, dt,
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_2}^{R_2} k^{+}(r,t) \, dr \, dt.
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_2}^{R_2} k^{+}(r,t) \, dr \, dt.
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_2}^{R_2} k^{+}(r,t) \, dr \, dt.
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_2}^{R_2} k^{+}(r,t) \, dr \, dt.
$$
\n
$$
e^{\int_{R_1}^{R_2} k^{+}(r,t) \, dr} = \int_{R_2}
$$

 $\forall$   $t\in[R_1,R_2]$ ,  $\omega\in S^{\scriptscriptstyle N-1};$  and  $^CW_q^+(t\omega)$  is integrable over  $\big[R_1,R_2\big]$ ,  $\forall$   $\omega\in S^{\scriptscriptstyle N-1}.$ 

Here  $\Phi$  :  $R_{+}^{n}$   $\rightarrow$   $R$  is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (145) follows:

Theorem 47 All as in Remark 46. Then

Vectorial Prabhakar Hardy Type Generalized Fractional Inequalities

\nHere 
$$
\Phi: \mathbb{R}_+^n \to \mathbb{R}
$$
 is a convex and increasing per coordinate function.

\nA direct application of Theorem 42, along with (145) follows:

\n**Theorem 47** All as in Remark 46. Then

\n
$$
\int_A^C L_q^*(x) \Phi\left(\frac{\left|\frac{\binom{C}{R}D_{\rho,\mu,\nu,R_1+}^{x,y}f}{\binom{C}{x}}\right|}{\binom{C}{x}(x)}\right) dx \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_A^C W_q^*(x) \Phi\left(\frac{\left|\frac{\overline{f}_\nu^{[N]}(x)}{\binom{C}{y}}\right|}{\left|\frac{\overline{f}_\nu^{[N]}(x)}{\binom{C}{y}}\right|}\right) dx,
$$
\nwhere  $\left(\frac{C}{R}D_{\rho,\mu,\nu,R_1+}^{x,y,R_1+}f\right)(x) = \left(\frac{C}{R}D_{\rho,\mu,\nu,R_1+}^{x,y,R_1+}f_1(x),\ldots,\left(\frac{C}{R}D_{\rho,\mu,\nu,R_1+}^{x,y,R_1+}f_n(x)\right)\right)$  and the coordinates are assumed to be continuous on  $\overline{A}$ .

\nWe make

 $\,$  $w, R_1 + J \uparrow \bigwedge^{\mathcal{M}} J$  $\,$  $W, R_1 + J \sim \mathcal{N}$  $\,$  $\binom{C}{R}D^{\gamma;\psi}_{\rho,\mu,w,R_1+}f(x) = \binom{C}{R}D^{\gamma;\psi}_{\rho,\mu,w,R_1+}f_1(x),..., \binom{C}{R}D^{\gamma;\psi}_{\rho,\mu}$  $\rho,\mu$  $\gamma$  ;  $\psi$  $\rho,\mu$  $\gamma;\psi$  $\rho,\mu$ ;  $1 \mathcal{N}$   $\mathcal{N}$   $\cdots$   $\mathcal{N}$   $\mathcal{N}_{\rho,\mu,w,R_1}$ ;  $, \mu, w, R_1$ ;  $\Gamma^{\mu}_{\mu,\mu,w,R_1+}f(x)\!=\! \binom{C}{k} \!\!D^{\nu;\nu}_{\rho,\mu,w,R_1+}f_1\big(x),\!..., \left(\!\!\!\begin{array}{c}C\!\! D^{\nu;\nu}_{\rho,\mu,w,R_1+}f_n\big(x)\big)\!\!\end{array}\right)$  and the coordinates are assumed to be continuous on  $A$ .

We make

**Remark 48** Let  $\rho,\mu,w > 0, \, \gamma < 0$  ,  $N = \lceil \mu \rceil, \quad \mu \notin \mathsf{N};\, \ f_i \in C^\mathbb{N}\big(\overline{A}\big)$   $i=1,...,n,$  and  $\overrightarrow{f} = (f_1,...,f_n),$  and  $\psi \in C^N([R_1, R_2])$ ,  $\psi'(r) \neq 0$ ,  $\forall r \in [R_1, R_2]$  and  $\Psi$  is increasing. We follow Definition 45 and we set:

A direct application of Theorem 42, along with (145) follows:  
\n**Theorem 47** All as in Remark 46. Then  
\n
$$
\int_{A}^{C} L_{q}^{r}(x) \Phi\left(\frac{\sqrt{\sum_{k}^{T/2} D_{\rho,\mu,\nu,\mu_{1}}^{r}(f)}\sqrt{\sum_{k}^{r}}}{C L_{q}^{r}(x)}\right) dx \leq \left(\frac{R_{2}}{R_{1}}\right)^{N-1} \int_{A}^{c} W_{q}^{r}(x) \Phi\left(\frac{\sqrt{\sum_{k}^{T/2} (x)}\sqrt{\sum_{k}^{T/2} (x)}\sqrt{\sum_{k
$$

One can write that

$$
\left\| \overline{f_{\psi}^{[N]}(y)} \right\|_{q} = \left\| \overline{f_{\psi}^{[N]}(t\omega)} \right\|_{q}, 1 \leq q \leq \infty,
$$
\n
$$
\omega.
$$
\n<math display="block</math>

where  $t\in[R_{1},R_{2}]$  ,  $\omega\in S^{N-1}$  ;  $\mathcal{Y}=t\omega$  .

q  $C_{\psi}^{[N]}(y)$   $\in C(\overline{A}), 1 \leq q \leq \infty.$ q  $\|f_{\psi}^{[N]}(y)\|>0$  on  $\overline{A}$  ,  $1\!\leq\!q\!\leq\!\infty$  fixed.

Consider the kernel

$$
{}^{C}k^{-}(r,t):=k(r,t):=\mathcal{X}_{r,R_{2}}(t)\psi'(t)(\psi(t)-\psi(r))^{N-\mu-1}E_{\rho,N-\mu}^{-\gamma}\bigg[\psi(\psi(t)-\psi(r))^{\rho}\bigg]
$$
\n(156)

Let

$$
{}^{C}L_{q}^{-}(x) = {}^{C}L_{q}^{-}(r\omega) = \int_{R_{1}}^{R_{2}} {}^{C}k^{-}(r,t) \left\| \overline{f_{\psi}^{[N]}(t\omega)} \right\|_{q} dt,
$$
\n(157)

 $x \,{=}\, r \omega \,{\in}\, \overline{A}$  ,  $1 \,{\leq}\, q \,{\leq}\, \infty$  fixed;  $r \,{\in}\, [R_{\rm l},R_{\rm 2}]$  ,  $\omega \,{\in}\, S^{\scriptscriptstyle N-1}.$ 

We have that  ${}^{C}L_{q}^{-}(r\omega)$   $>$   $0$  for  $r$   $\in$   $(R_{1}, R_{2}]$ ,  $\forall$   $\omega$   $\in$   $S^{N-1}.$ 

Here we choose the weight  $u(x) \!= u(r\omega) \!=\!~^C L_q^-(r\omega).$ 

Consider the function

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\n• 0 for 
$$
r \in (R_1, R_2]
$$
,  $\forall \omega \in S^{N-1}$ .  
\n
$$
\text{ght } u(x) = u(r\omega) = {}^{C}L_q^-(r\omega).
$$
\n
$$
{}^{C}W_q^-(y) = {}^{C}W_q^-(t\omega) = \left\| \overline{f_{\psi}^{[N]}(t\omega)} \right\|_q \left( \int_{R_1}^{R_2} {}^{C}k^-(r,t)dr \right) < \infty,
$$
\n
$$
\text{and } {}^{C}W_q^-(t\omega) \text{ is integrable over } [R_1, R_2], \forall \omega \in S^{N-1}.
$$
\n(158)

 $\forall$   $t\in[R_1,R_2]$ ,  $\omega\in S^{N-1};$  and  $^CW_q^-(t\omega)$  is integrable over  $\big[R_1,R_2\big]$ ,  $\forall$   $\omega\in S^{N-1}.$ 

Here  $\Phi$  :  $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}$  is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (147) follows:

Theorem 49 All as in Remark 48. Then

We have that 
$$
{}^{C}L_{q}(r\omega) > 0
$$
 for  $r \in (R_{1}, R_{2}]$ ,  $\forall \omega \in S^{N-1}$ .  
\nHere we choose the weight  $u(x) = u(r\omega) = {}^{C}L_{q}(r\omega)$ .  
\nConsider the function  
\n
$$
{}^{C}W_{q}^{-}(y) = {}^{C}W_{q}^{-}(t\omega) = \left| \int_{r_{q}}^{r_{q}} V(t\omega) \right|_{q} \left( \int_{R_{1}}^{R_{2}} \mathcal{E}_{r} (r, t) dr \right) < \infty, \qquad (158)
$$
\n
$$
\forall t \in [R_{1}, R_{2}], \omega \in S^{N-1}; \text{ and } {}^{C}W_{q}^{-}(t\omega) \text{ is integrable over } [R_{1}, R_{2}], \forall \omega \in S^{N-1}.
$$
\nHere  $\Phi : R_{1}^{n} \rightarrow \mathbb{R}$  is a convex and increasing per coordinate function.  
\nA direct application of Theorem 44, along with (147) follows:  
\n**Theorem 49** All as in Remark 48. Then  
\n
$$
\int_{A} {}^{C}I_{q}^{-}(x) \Phi\left( \frac{\sqrt{\sum_{i} p_{j_{i},k_{i},R_{2}}^{N}}}{\sum_{i} f_{i}(x)} \right) dx \leq \left( \frac{R_{2}}{R_{1}} \right)^{N-1} \int_{A} {}^{C}W_{q}^{-}(x) \Phi\left( \frac{\sqrt{\int_{r_{i}}^{R_{1}} V_{i}(x)}}{\int_{r_{i}}^{R_{2}} V_{i}(x)} \right) dx, \qquad (159)
$$
\nwhere  $(\overline{\sum_{i} p_{j_{i},k_{i},R_{i}-1}^{N}} f(x) = ((\sum_{i} p_{j_{i},k_{i},R_{i}-1}^{N}) f(x),...,( \sum_{i} p_{j_{i},k_{i},R_{i}-1}^{N}) f(x))$  and the coordinates are assumed to be continuous on  $\overline{A}$ .  
\nWe need  
\n**Definition 50** Let  $P, H, W > 0, \gamma < 0$ ,  $N = [ \mu ]$ ,  $\mu \notin \mathbb{N}$ ,  $f \in C(\overline{A})$  and  $\psi \in C^{X}([R_{1}, R_{2}], \psi(r) \neq$ 

 $\,$  $w, R_2$  –  $J_1 \wedge N_2$   $\cdots$ ,  $\langle R \rangle$  $\,$  $W, R_2$  -  $\bigcup$   $\mathcal{N}$   $\bigcup$   $R$  $\,$  $\binom{C}{R}D^{\gamma,\psi}_{\rho,\mu,w,R_2}f(x) = \binom{C}{R}D^{\gamma,\psi}_{\rho,\mu,w,R_2}f_1(x),...,\binom{C}{R}D^{\gamma,\psi}_{\rho,\mu}$  $\rho,\mu$  $\gamma$  ;  $\psi$  $\rho,\mu$  $\gamma$ ; $\psi$  $\rho,\mu$ ;  $_1 \mathcal{N}$   $\mathcal{N}$   $\cdots$   $\mathcal{N}$   $\mathcal{N$ ;  $, \mu, w, R_2$ ;  $L^{\mu\nu}_{\mu,\mu,w,R_2-}f\big(x\big)=\big(\!\!\big(\!\!\big(\!{}^C_R D^{Z;\psi}_{\rho,\mu,w,R_2-}f_1\big)\!\!\big(x\!\!\big),\!...\!,\!{C}_R D^{Z;\psi}_{\rho,\mu,w,R_2-}f_n\big)\!\!\big(x\big)\!\!\big)$  and the coordinates are assumed to be continuous on  $A$ .

We need

Definition 50 Let  $\rho,\mu,w \geq 0, \ \gamma < 0$  ,  $N = \lceil \mu \rceil, \ \ \mu \notin \mathsf{N};\ f \in C\overline{(A)}$  and  $\psi \in C^{\scriptscriptstyle N}([R_{\scriptscriptstyle 1},R_{\scriptscriptstyle 2}]), \ \psi^{'}(r) \neq 0$  ,  $\forall$  $r\!\in\!\!{[R_1,R_2]}$  and  $\mathscr V$  is increasing. The  $\mathscr V$  -Prabhakar-Riemann Liouville left and right radial fractional derivatives of order  $\mu$  are defined as follows (see also Definition 40)  $\frac{1}{\binom{n}{2}-f(x)} = \left( \binom{n}{k} p_{\rho,k,w,k_2}^{rw} f_1(x), \dots, \binom{n}{k} p_{\rho,k,w,k_2}^{rw} f_n(x) \right)$  and the coordinates are assumed to be<br>  $0$  Let  $P, \mu, w > 0, \gamma < 0$ ,  $N = \lceil \mu \rceil, \mu \notin \mathbb{N}, f \in C(\overline{A})$  and  $\psi \in C^{\times}(\lceil R_1, R_2 \rceil), \psi'(r) \neq 0$ ,  $\forall$ 

$$
\left(\,_{R}^{RL} D_{\rho,\mu,w,R_{1}+}^{\gamma;\psi} f\right)\! (x) = \left(\,_{R}^{RL} D_{\rho,\mu,w,R_{1}+}^{\gamma;\psi} f\right)\! (r\omega) := \quad \left(\frac{1}{\psi^{'}(r)} \frac{d}{dr}\right)^{N} \left(e_{\rho,N-\mu,w,R_{1}+}^{-\gamma;\psi} f\right)\! (x),\tag{160}
$$

and

$$
\left(\begin{array}{c}\n\frac{RL}{R}D_{\rho,\mu,w,R_2}^{\gamma;\nu} - f(x)\n\end{array}\right) = \left(\begin{array}{c}\n\frac{RL}{R}D_{\rho,\mu,w,R_2}^{\gamma;\nu} - f(x)\n\end{array}\right) = -\left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^N \left(e_{\rho,N-\mu,w,R_2}^{-\gamma;\nu} - f(x)\n\right),\n\tag{161}
$$

 $\forall x\in \overline{A}; \text{ where } x\mathop{=}\limits r\varpi \text{ , } r\in \left[ R_1,R_2 \right] \text{, } \varpi \in S^{N-1} \text{.}$ 

In this work we assume that  $\binom{RL}{R} \! D^{ \gamma; \psi}_{\rho,\mu, w,R_1} f \Bigr), \, \binom{RL}{R} \! D^{ \gamma; \psi}_{\rho,\mu, w,R_2} f \Bigl) \! \in C \bigl( A$  $_{R}^{\scriptscriptstyle KL}D^{\scriptscriptstyle\gamma;\psi}_{\rho,\mu}$  $\rho,\mu$ ;  $\left(\begin{matrix}C_{k,m}^{W}\\ \mu_{k,m}R_{1}+f\end{matrix}\right),\left(\begin{matrix}RL & D^{\gamma,\psi}_{\rho,\mu,\psi,R_{2}-}f\end{matrix}\right)\in C(\overline{A}).$  $\rho,\mu$ ;  $C_{\mu,\mu,w,R_2}^{\nu}$ - $f$   $\in C(A)$ .

Next we define the  $\psi$ -Hilfer-Prabhakar left and right radial fractional derivatives of order  $\mu$  and type  $\beta \in [0,1]$ , as follows ( $\xi := \mu + \beta\big(N - \mu\big)$ , see also Definition 40):

$$
\left({}^{H}_{R}D^{\gamma,\beta;\psi}_{\rho,\mu,w,R_{1}+}f\right)(x) = \left({}^{H}_{R}D^{\gamma,\beta;\psi}_{\rho,\mu,w,R_{1}+}f\right)(r\omega) := e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,w,R_{1}+} \left({}^{RL}_{R}D^{\gamma(1-\beta);\psi}_{\rho,\xi,w,R_{1}+}f\right)(x),
$$
\n(162)

and

$$
\left({}^{H}_{R}D^{\gamma,\beta;\psi}_{\rho,\mu,\nu,R_{2}}f\right)(x) = \left({}^{H}_{R}D^{\gamma,\beta;\psi}_{\rho,\mu,\nu,R_{2}}f\right)(r\omega) := e^{-\gamma\beta;\psi}_{\rho,\xi-\mu,\nu,R_{2}}\left({}^{RL}_{R}D^{\gamma(1-\beta);\psi}_{\rho,\xi,\nu,R_{2}}f\right)(x),
$$
\n(163)

 $\forall x\in \overline{A}; \text{ where } x\mathop{=}\limits r\varpi \text{ , } r\in \left[ R_1,R_2 \right] \text{, } \varpi \in S^{N-1} \text{.}$ 

In this work we assume that 
$$
\binom{H}{R} \mathbf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi} f
$$
,  $\binom{H}{R} \mathbf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi} f$   $\in C(\overline{A})$ .

We make

Remark 51 Let  $\rho,\mu,w > 0,$   $\gamma < 0$  ,  $N = \lceil \mu \rceil$ ,  $\mu \notin \mathsf{N};$   $0 \leq \beta \leq 1$  ,  $\xi = \mu + \beta(N-\mu)$ ,  $\int_{i} \in C(\overline{A})$   $i = 1,...,n,$  and  $\psi \in C^N([R_1, R_2])$ ,  $\psi'(r) \neq 0$ ,  $\forall r \in [R_1, R_2]$ , and  $\Psi$  is increasing. We follow Definition 50, especially (162) and we set:

$$
\begin{pmatrix}\n\frac{m}{N}D_{\rho,\mu,\nu,\mu_{1}}^{\gamma,\beta\psi}f\left(x\right) = \left(\frac{m}{N}D_{\rho,\mu,\nu,\mu_{1}}^{\gamma,\beta\psi}f\left(m\right)\right) = e_{\rho,\xi-\mu,\nu,\mu_{1}}^{\gamma,\beta\psi}\left(\frac{m}{N}D_{\rho,\xi,\nu,\mu_{1}}^{\gamma(1-\mu)\mu}\right)f\left(x\right)
$$
\n(162)  
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\n1  
\n1  
\n1  
\n1  
\n1  
\n2  
\n2  
\n3  
\n4  
\n
$$
\begin{pmatrix}\n\frac{m}{N}D_{\rho,\mu,\nu,\mu_{2}}^{\gamma,\beta\psi}f\left(x\right) = \left(\frac{m}{N}D_{\rho,\mu,\nu,\mu_{2}}^{\gamma,\beta\psi}f\right)f\left(\rho\right) := e_{\rho,\xi-\mu,\nu,\mu_{2}}^{\gamma,\beta\psi}f\left(\frac{m}{N}D_{\rho,\xi,\mu,\mu_{2}}^{\gamma(1-\mu)\mu}\right)f\left(x\right)
$$
\n(163)  
\n
$$
x \in \overline{A}; \text{ where } x = r\omega, r \in [R_{1}, R_{2}] \omega \in S^{N-1}.
$$
\n10 this work we assume that 
$$
\begin{pmatrix}\n\frac{m}{N}D_{\rho,\mu,\nu,\mu_{1}}^{\gamma,\beta\psi}f\left(x\right) & \left(\frac{m}{N}D_{\rho,\mu,\nu,\mu_{2}}^{\gamma,\beta\psi}f\right) \in C(\overline{A}).\n\end{pmatrix}
$$
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One can write that

$$
\left\| \frac{\left(\frac{RL}{R} D_{\rho,\xi,w,R_1+f}^{(1-\beta),\psi} \right)}{\left(\frac{L}{R} D_{\rho,\xi,w,R_1+f}^{(1-\beta),\psi} \right)} \right\|_q = \left\| \frac{\left(\frac{RL}{R} D_{\rho,\xi,w,R_1+f}^{(1-\beta),\psi} \right)}{\left(\frac{L}{R} D_{\rho,\xi,w,R_1+f}^{(1-\beta),\psi} \right)} \right\|_q, 1 \leq q \leq \infty,
$$
\n(165)  
\n
$$
\left\| \frac{L}{R} \left( \frac{L}{R} \right) \right\|_q < O \text{ on } \overline{A}, 1 \leq q \leq \infty \text{ fixed.}
$$
\n
$$
= k(r,t) := \chi_{(R_1,r]}(t) \psi'(t) \left(\psi(r) - \psi(t)\right)^{\varepsilon-\mu-1} E_{\rho,\xi-\mu}^{-\gamma\beta} \left[ w(\psi(r) - \psi(t))^{\rho} \right] \tag{166}
$$
\n
$$
= \sum_{k=1}^{\infty} k(r,t) \left( \frac{R}{R} \sum_{k=1}^{\infty} \frac{L}{R} \psi(k) \right)^{\varepsilon-\mu-1} \left( \frac{R}{R} \sum_{k=1}^{\infty} \frac{L}{R} \psi(k) \right)^{\varepsilon-\mu} \left( \frac{L}{R} \sum_{k=1}^{\infty} \frac{L}{R} \psi(k) \right)^{\varepsilon-\mu} \frac{1}{r} \tag{167}
$$
\n
$$
= \frac{1}{2} \sum_{k=1}^{\infty} \frac{L}{R} \psi(k) e^{-\mu k} \left( \frac{L}{R} \sum_{k=1}^{\infty} \frac{L}{R} \psi(k) \right)^{\varepsilon-\mu-1} \frac{1}{r} \left( \frac{L}{R} \sum_{k=1}^{\infty} \frac{L}{R} \psi(k) \right)^{\varepsilon-\mu} \frac{1}{r} \tag{167}
$$

where  $t \in [R_1, R_2]$ ,  $\omega \in S^{N-1}$ ;  $y = t\omega$ .

 $w, R_1 + J \left\Vert \mathcal{N} \right\Vert_q$  $\left\|R^L D^{\gamma(1-\beta);\psi}_{\rho,\xi,w,R_1+} f\right\|y\right\| \in C$ γ(1–β);ψ<br>ρ,ζ,w,R<sub>1</sub> +.  $\rho,\xi$  $1-\beta$ ;  $\left\lbrack\begin{array}{l} (1-\beta)\mathbf{1} & \mathbf{1} \ \mathbf{1} & \mathbf{1} \ \mathbf{1} & \mathbf{1} \end{array}\right\rbrack_{\mathbb{R}}, \mathbf{1} & \mathbf{1} \mathbf{1} \mathbf{1} \mathbf{1} \leq q \leq \infty.$ 

 $1-\beta$  );  $\left\Vert \xi,w,R_{1}+J\right\Vert \mathcal{N}$  $\frac{R}{R}D^{\gamma(1-\beta);\psi}_{\rho,\xi,w,R_1+}f\Big(y\Big)$  $\rho,\xi$  $\overline{a}$  $\Vert f \big({\mathcal Y} \big) \Vert \geq 0$  on  $A$  ,  $1 \leq q \leq \infty$  fixed.

Consider the kernel

$$
{}^{P}k^{+}(r,t):=k(r,t):=\chi_{(R_{1},r]}(t)\psi^{'}(t)(\psi(r)-\psi(t))^{\xi-\mu-1}E_{\rho,\xi-\mu}^{-\gamma\beta}\big[\psi(\psi(r)-\psi(t))^{\rho}\big]
$$
\n(166)

Let

$$
{}^{P}L_{q}^{+}(x) = {}^{P}L_{q}^{+}(r\omega) = \int_{R_{1}}^{R_{2}} {}^{P}k^{+}(r,t) \left\| \left( \frac{RL}{R} D_{\rho,\xi,w,R_{1}}^{\gamma(1-\beta),\psi} f \right) (t\omega) \right\|_{q} dt,
$$
\n(167)

 $x = r\omega \in \overline{A}$ ,  $1 \le q \le \infty$  fixed;  $r \in [R_1, R_2]$ ,  $\omega \in S^{N-1}$ . We have that  ${}^{{}^P}L_q^{\!+}(r\omega)\!>\!0$  for  $r\in (R_{\!1},R_{\!2}], \,\forall\;\;\omega\!\in\! S^{N-\!1}.$ Here we choose the weight  $u(x) \!= u(r\omega) \!= {}^P L^+_q\!(r\omega)$ .

Consider the function

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\nfixed; 
$$
r \in [R_1, R_2]
$$
,  $\omega \in S^{N-1}$ .  
\n $\omega$ )>0 for  $r \in (R_1, R_2]$ ,  $\forall \omega \in S^{N-1}$ .  
\n $\Rightarrow$  weight  $u(x) = u(r\omega) = {}^P L_q^*(r\omega)$ .  
\non  
\n ${}^P W_q^+(y) = {}^P W_q^*(t\omega) = \left\| \begin{pmatrix} \frac{RL}{R} D_{\rho,\xi,w,R_1}^{y(1-\beta),\psi} f(t\omega) \\ \frac{RL}{R} D_{\rho,\xi,w,R_1}^{y(1-\beta),\psi} f(t\omega) \end{pmatrix} \right\|_q \left( \int_{R_1}^{R_2} k^+(r,t) dr \right) < \infty$ , (168)

 $\forall$   $t\in[R_1,R_2]$ ,  $\omega\in S^{N-1};$  and  ${}^PW_q^+(t\omega)$  is integrable over  $\big[R_1,R_2\big]$ ,  $\forall$   $\omega\in S^{N-1}.$ 

Here  $\Phi$  :  $\mathsf{R}^n_+$   $\rightarrow$   $\mathsf{R}$  is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (162) follows:

Theorem 52 All as in Remark 51. Then

We have that 
$$
{}^{P}L_{q}^{*}(x\omega) > 0
$$
 for  $r \in (R_{1}, R_{2}], \forall \omega \in S^{N-1}.$   
\nHere we choose the weight  $u(x) = u(r\omega) = {}^{P}L_{q}^{*}(r\omega).$   
\nConsider the function  
\n
$$
{}^{P}W_{q}^{+}(y) = {}^{P}W_{q}^{+}(t\omega) = \left\| \frac{W_{\alpha}D_{\rho,\xi,w,R_{1}}^{*}(F_{\alpha})}{\left\| R_{\alpha}D_{\rho,\xi,w,R_{1}}^{*}(F_{\alpha}) \right\|_{q}} \right\|_{q} \left\{ \int_{R_{1}}^{R_{2}} \int_{R_{1}}^{R_{2}} k^{+}(r,t) dr \right\} < \infty,
$$
\n(168)  
\n $\forall t \in [R_{1}, R_{2}], \omega \in S^{N-1};$  and  ${}^{P}W_{q}^{+}(t\omega)$  is integrable over  $[R_{1}, R_{2}], \forall \omega \in S^{N-1}.$   
\nHere  $\Phi : R_{+}^{n} \rightarrow R$  is a convex and increasing per coordinate function.  
\nA direct application of Theorem 42, along with (162) follows:  
\n**Theorem 52** All as in Remark 51. Then  
\n
$$
\int_{A} {}^{P}L_{q}^{+}(x) \Phi\left( \frac{\left\| \frac{W_{\alpha}D_{\rho,\mu,w,R_{1}}^{*}(F_{\alpha})}{\left\| R_{\rho,\mu,w,R_{1}}^{*}(F_{\alpha}) \right\|_{q}} \right\| dx \leq \left( \frac{R_{2}}{R_{1}} \right)^{N-1} \int_{A} {}^{P}W_{q}^{+}(x) \Phi\left( \frac{\left\| \frac{W_{\alpha}D_{\rho}^{*}(t-\beta)W_{\rho}^{*}(F_{\alpha})}{\left\| R_{\rho,\mu,w,R_{1}}^{*}(F_{\alpha}) \right\|_{q}} \right\| dx, \qquad (169)
$$
\nwhere  $\frac{W_{\alpha}D_{\rho,\mu,w,R_{1}}^{*}f_{\alpha}(x)}{\left\| R_{\rho,\mu,w,R_{1}}^{*}(x)f_{\alpha}\right\|_{q}} = \left( \frac{W_{\alpha}D_{\rho,\mu,w,R_{1}}^{*}f_{\alpha}(x)}{\left\| R_{\rho,\mu$ 

H  $w, R_1 + J \uparrow \bigwedge^{\mathcal{M}} J$ H  $W, R_1 + J \sim \mathcal{N}$ H  $\binom{H}{R} \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_1+} f\left(\!\!\left(\begin{smallmatrix}X\\ \end{smallmatrix}\right.\!\!\right) = \left(\!\!\left(\!\!\begin{smallmatrix}H\\ R\end{smallmatrix}\right.\!\!\!\right. \!\! \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_1+} f_1\left(\!\!\left(\begin{smallmatrix}X\\ \end{smallmatrix}\right.\!\!\right),..., \left(\!\!\begin{smallmatrix}H\\ R\end{smallmatrix}\right.\!\!\!\right. \!\! \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_1+} f_1\left(\!\!\left(\begin{smallmatrix}X\\ \end{$  $\rho, \mu$  $\gamma, \beta; \psi$  $\rho,\mu$  $\gamma, \beta; \psi$  $\rho,\mu$  $,\beta;$  $_1 \mathcal{N}$   $\mathcal{N}$   $\cdots$   $\mathcal{N}$   $\mathcal{N$  $,\beta;$  $, \mu, w, R_1$  $,\beta;$  $\mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_1+}f\left(\!\! \left\langle x\right. \!\!\right\rangle \!\!=\!\left(\!\! \left\langle \!\! \left\langle R\mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_1+}f_1 \right\rangle \!\! \left\langle x\right. \!\! \right\rangle \!\!,..., \left\langle \!\! \left\langle \!\! \left\langle R\mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_1+}f_n \right\rangle \!\! \left\langle x\right. \!\! \right\rangle \!\!\right)$  and the coordinates are assu continuous on  $A$ .

We make

Remark 53 Let  $\rho,\mu,w \geq 0,$   $\gamma < 0$  ,  $N = \lceil \mu \rceil$ ,  $\mu \notin \mathsf{N};$   $0 \leq \beta \leq 1$  ,  $\xi = \mu + \beta(N-\mu),$   $\int_{i} \in C(\overline{A})$   $i = 1,...,n,$  and  $\psi \in C^N([R_1, R_2])$ ,  $\psi'(r) \neq 0$ ,  $\forall r \in [R_1, R_2]$ , and  $\Psi$  is increasing. We follow Definition 50, especially (163) and we set:

in Remark 51. Then  
\n
$$
\Phi\left[\frac{\left(\frac{H}{R}D_{\rho,\mu,w,R_{1}+f}^{N}\right)(x)}{P_{L_{q}}(x)}\right]dx \leq \left(\frac{R_{2}}{R_{1}}\right)^{N-1} \int_{d}^{p} W_{q}^{+}(x)\Phi\left[\frac{\left|\frac{R_{L}}{R}D_{\rho,\xi,w,R_{1}+f}^{N}\right)(x)}{\left|\frac{R_{L}}{R}D_{\rho,\xi,w,R_{1}+f}^{N}\right)(x)}\right|_{q}dx, \qquad (169)
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(191
$$

One can write that

$$
\left\| \left( \mathcal{R}^L D_{\rho,\xi,w,R_2}^{\gamma(1-\beta); \psi} f \right)_{q} \right\|_{q} = \left\| \left( \mathcal{R}^L D_{\rho,\xi,w,R_2}^{\gamma(1-\beta); \psi} f \right)_{q}, 1 \le q \le \infty, \tag{171}
$$

Vectorial Prahahkar Hardy Type Generalized Fractional Inequalities  
\nwhere 
$$
t \in [R_1, R_2]
$$
,  $\omega \in S^{N-1}$ ;  $y = t\omega$ .  
\nNotice that  $\left\| \begin{pmatrix} \frac{R_L}{R} D_{\rho,\xi,w,R_2}^{y(1-\beta)k\psi} - f(y) \end{pmatrix} \right\|_q \in C(\overline{A})$ ,  $1 \le q \le \infty$ .  
\nWe assume that  $\left\| \begin{pmatrix} \frac{R_L}{R} D_{\rho,\xi,w,R_2}^{y(1-\beta)k\psi} - f(y) \end{pmatrix} \right\|_q > 0$  on  $\overline{A}$ ,  $1 \le q \le \infty$  fixed.  
\nConsider the Kernel  
\n $P_K^-(r,t) := k(r,t) := \chi_{[r,R_2)}(t) \psi'(t) (\psi(t) - \psi(r))^{z-\mu-1} E_{\rho,\xi-\mu}^{-y\beta} [\psi(\psi(t) - \psi(r))^{\rho}]$  (172)  
\nLet  
\n $P_{\overline{L_q}}(x) =^p L_q^-(r\omega) = \int_{R_1}^{R_2} k^-(r,t) \left\| \begin{pmatrix} \frac{R_L}{R} D_{\rho,\xi,w,R_2}^{y(1-\beta)k\psi} - f(t\omega) \end{pmatrix} \right\|_q dt$ , (173)  
\n $x = r\omega \in \overline{A}$ ,  $1 \le q \le \infty$  fixed;  $r \in [R_1, R_2]$ ,  $\omega \in S^{N-1}$ .

Consider the kernel

$$
{}^{P}k^{-}(r,t):=k(r,t):=\chi_{[r,R_{2})}(t)\psi'(t)(\psi(t)-\psi(r))^{\xi-\mu-1}E_{\rho,\xi-\mu}^{-\gamma\beta}\big[\psi(\psi(t)-\psi(r))^{\rho}\big]
$$
\n(172)

Let

$$
{}^{P}L_{q}^{-}(x) = {}^{P}L_{q}^{-}(r\omega) = \int_{R_{1}}^{R_{2}} {}^{P}k^{-}(r,t) \left\| \left( {}^{RL}_{R}D_{\rho,\xi,w,R_{2}}^{\gamma(1-\beta),\psi} - f \right) \left( t\omega \right) \right\|_{q} dt,
$$
\n(173)

 $x \,{=}\, r \omega \,{\in}\, \overline{A}$  ,  $1 \,{\leq}\, q \,{\leq}\, \infty$  fixed;  $r \,{\in}\, [R_{\rm l},R_{\rm 2}]$  ,  $\omega \,{\in}\, S^{\scriptscriptstyle N-1}.$ We have that  ${}^{{}^{{}_{P}}L_{q}^{-}}(r\omega)$   $>$   $0$  for  $r$   $\in$   $(R_{_{1}}, R_{_{2}}], \forall$   $\omega$   $\in$   $S^{N-1}.$ 

Here we choose the weight  $u(x)$  =  $u(r\omega)$  =  $^PL_q^-(r\omega)$  .

Consider the function

$$
(\cdot) := k(r, t) := \chi_{[r, R_2)}(t) \psi'(t) (\psi(t) - \psi(r))^{\xi - \mu - 1} E_{\rho, \xi - \mu}^{-\gamma \beta} [\psi(\psi(t) - \psi(r))^\rho]
$$
(172)  
\n
$$
{}^P L_q^-(x) = {}^P L_q^-(r\omega) = \int_{R_1}^{R_2} k^-(r, t) \left\| \frac{\kappa}{\kappa} D_{\rho, \xi, w, R_2}^{(1-\beta)\psi} - f(t\omega) \right\|_q dt,
$$
(173)  
\nfixed;  $r \in [R_1, R_2], \omega \in S^{N-1}.$   
\n $\omega$ ) > 0 for  $r \in (R_1, R_2], \forall \omega \in S^{N-1}.$   
\n $\omega$  weight  $u(x) = u(r\omega) = {}^P L_q^-(r\omega).$   
\non  
\n $P W_q^-(y) = {}^P W_q^-(t\omega) = \left\| \frac{\kappa}{\kappa} D_{\rho, \xi, w, R_2 - f}^{(1-\beta)\psi} f(t\omega) \right\|_q \left( \int_{R_1}^{R_2} k^-(r, t) dr \right) < \infty,$   
\n(174)  
\n $\int_{-1}^{1} k^2 \left( \frac{\kappa}{\kappa} \right)^{1/2} \left( \frac{\kappa}{\kappa} \right)^{1/2} \left( \int_{R_1}^{R_2} k^-(r, t) dr \right) < \infty,$ 

 $\forall$   $t\in[R_1,R_2]$ ,  $\omega\in S^{N-1};$  and  ${}^PW_q^-(t\omega)$  is integrable over  $\big[R_1,R_2\big]$ ,  $\forall$   $\omega\in S^{N-1}.$ 

Here  $\Phi: R^n_+ \to R$  is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (163) follows:

Theorem 54 All as in Remark 53. Then

We have that 
$$
{}^{P}L_{q}(x) \geq 0
$$
 for  $r \in (R_{1}, R_{2}], \forall \omega \in S^{N-1}$ .  
\nHere we choose the weight  $u(x) = u(r\omega) = {}^{P}L_{q}(r\omega)$ .  
\nConsider the function  
\n ${}^{P}W_{q}(y) = {}^{P}W_{q}(t\omega) = \left\| \frac{\overline{R}_{L}D_{\rho,\xi,w,R_{2}}^{v(1-\beta)p} f(x\omega)}{\overline{R}_{R}D_{\rho,\xi,w,R_{2}}^{p}(1-\beta)} \right\|_{q} \left\{ \int_{R_{1}}^{R_{2}} \overline{R}_{L}F(r,t)dr \right\} < \infty,$ \n $\forall t \in [R_{1}, R_{2}], \omega \in S^{N-1}$ ; and  ${}^{P}W_{q}(t\omega)$  is integrable over  $[R_{1}, R_{2}], \forall \omega \in S^{N-1}$ .  
\nHere  $\Phi: R_{+}^{n} \rightarrow R$  is a convex and increasing per coordinate function.  
\nA direct application of Theorem 44, along with (163) follows:  
\n**Theorem 54** All as in Remark 53. Then  
\n
$$
\int_{A} {}^{P}L_{q}(x)\Phi\left(\frac{\overline{R}_{L}D_{\rho,\mu,\mu,\rho}^{v(\beta)p} f(x)}{\overline{R}_{L}F_{Q}(x)}\right)dx \leq \left(\frac{R_{2}}{R_{1}}\right)^{N-1} \int_{A} {}^{P}W_{q}(x)\Phi\left(\frac{\overline{R}_{L}D_{\rho,\xi,\mu,\rho}^{v(1-\beta)p} f(x)}{\overline{R}_{L}D_{\rho,\xi,\mu,\rho,\rho}^{p}(1-\beta)p} f(x)\right)dx,
$$
\n(175)  
\nwhere  $\frac{\overline{R}_{L}D_{\rho,\mu,\mu,\rho,\rho}^{v(\beta)p} f(x)}{\overline{R}_{L}D_{\rho,\mu,\mu,\rho,\rho}^{p}(1-\beta)p} f(x), ..., \left(\frac{u}{R}D_{\rho,\mu,\mu,\rho,\rho}^{v(\beta)p} f(x)\right)$  and the coordinates are assumed to be continuous on  $\overline{A}$ .  
\nWe make

H  $w, R_2 - J_1 \uparrow \mathcal{M}$   $\mathcal{M}$   $\cdots$ ,  $\mathcal{M}$ H  $W, R_2$  -  $\bigcup$   $\mathcal{N}$   $\mathcal{N}$   $\bigcup$   $R$ H  $\binom{H}{R} \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_2} f(x) = \left( \binom{H}{R} \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,R_2} f_1(x), \ldots, \binom{H}{R} \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w} \right)$  $\rho,\mu$  $\gamma, \beta; \psi$  $\rho,\mu$  $\gamma, \beta; \psi$  $\rho,\mu$  $,\beta;$  $_1 \mathcal{N}$   $\mathcal{N}$   $\cdots$   $\mathcal{N}$   $\mathcal{N$  $,\beta;$  $, \mu, w, R_2$  $,\beta;$  $\mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,{R_2}-f}\big({\chi}\big) = \big(\! \textstyle \frac{H}{R} \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,{R_2}-f_1}\hspace{-1mm}\big({\chi}\big),\!..., \big(\textstyle \frac{H}{R} \mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,w,{R_2}-f_n}\hspace{-1mm}\big) \big\vert_{\lambda,\mu,\nu,\rho} \big)$  and the coordinates are assumed to be continuous on  $A$ .

We make

**Remark 55** Let 
$$
f_{ji} \in C([a,b])
$$
,  $j = 1,2; i = 1,...,m; \psi \in C^{1}([a,b])$  which is increasing. Let also  $\rho_{i}, \mu_{i}, \gamma_{i}, \omega_{i} > 0$ 

and  $\big\|e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_{ji}\big\|(x),\ x\in [a,b]$  as in (2). We  $\gamma$  ; ;  $\psi$  $\rho$  ,  $\mu$  ,  $\omega$ ;  $\psi^{\psi}_{\mu_1,\omega_1,a+\int_{J^i}}(x),\;x\!\in\!{[a,b]}$  as in (2). We assume here that  $0\!<\!f_{2i}(y)\!<\!\infty\,$  on  ${[a,b]}$ ,  $i\!=\!1,...,m.$ 

Here we consider the kernel

is  
\n
$$
f_{ji}(x), x \in [a, b]
$$
 as in (2). We assume here that  $0 < f_{2i}(y) < \infty$  on  $[a, b]$ ,  $i = 1, \ldots, m$ .

\non  
\n
$$
k_i^*(x, y) := k_i(x, y) = \n\begin{cases} \n\psi'(y)(\psi(x) - \psi(y))^{\mu_i - 1} E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i(\psi(x) - \psi(y))^{\rho_i}] \, a < y \leq x, \\
0, x < y < b, \n\end{cases}
$$
\n

\nis  
\n
$$
w_i(y) := f_{2i}(y) \int_y^b u(x) \frac{k_i^*(x, y)}{\left(e_{\rho_i, \mu_i, \omega_i, a + f_{2i}}(x)\right)} dx < \infty,
$$
\n

\nand that  $\psi_i$  is integrable on  $[a, b]$ ,  $i = 1, \ldots, m$ .

\nis immediately implies:

\n(177)

 $i = 1,...,m$ .

Choose weight  $u(x) \ge 0$  , so that

$$
\psi_i(y) := f_{2i}(y) \int_y^b u(x) \frac{k_i^+(x, y)}{\left(e_{\rho_i, \mu_i, \omega_i, a+}^{y_i, y} f_{2i}(x)\right)} dx < \infty,
$$
\n(177)

a.e. on  $\big[a,b\big]$ , and that  $\psi_i$  is integrable on  $\big[a,b\big]$ ,  $i=1,...,m.$ 

Theorem 9 immediately implies:

**Theorem 56** All as in Remark 55. Let  $|p_i>1$  :  $\sum_{i=1}^{m} \frac{1}{n_i}=1$  .  $=$ 1  $P_i$ m i  $p_i > 1$  :  $\sum_{i=1}^n \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex and increasing. Then

$$
[0, x < y < b,
$$
  
\n
$$
\psi_{i}(y) := f_{2i}(y) \int_{y}^{b} u(x) \frac{k_{i}^{+}(x, y)}{\left(e_{\rho_{i}, \mu_{i}, \varphi_{i}, a}, f_{2i}\right) x} dx < \infty,
$$
\n(177)  
\n
$$
\psi_{i} \text{ is integrable on } [a, b], i = 1, ..., m.
$$
\n
$$
\text{ediately implies:}
$$
\nas in Remark 55. Let } p\_{i} > 1: \sum\_{i=1}^{m} \frac{1}{p\_{i}} = 1. \text{ Let the functions } \Phi\_{i}: \mathbb{R}\_{+} \to \mathbb{R}\_{+}, i = 1, ..., m, \text{ be convex}\n
$$
\int_{a}^{b} u(x) \prod_{i=1}^{m} \Phi_{i} \left( \frac{\left| \left(e_{\rho_{i}, \mu_{i}, \varphi_{i}, a}, f_{1i}\right) x \right|}{\left(e_{\rho_{i}, \mu_{i}, \varphi_{i}, a}, f_{2i}\right) x} \right) dx \leq \prod_{i=1}^{m} \left( \int_{a}^{b} \psi_{i}(y) \Phi_{i} \left( \frac{\left|f_{1i}(y)\right|}{f_{2i}(y)} \right)^{p_{i}} dy \right)^{\frac{1}{p_{i}}}.
$$
\n
$$
f_{ii} \in C([a, b]), \quad j = 1, 2; i = 1, ..., m; \quad \psi \in C^{1}([a, b]) \text{ which is increasing. Let also } \rho_{i}, \mu_{i}, \gamma_{i}, \omega_{i} > 0
$$

We make

**Remark 57** Let  $f_{ji}\in C([a,b])$ ,  $j=1,2;$   $i=1,...,m;$   $\psi\in C^{1}([a,b])$  which is increasing. Let also  $\rho_{i},\mu_{i},\gamma_{i},\omega_{i}>0$ and  $\big\|e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_{\scriptscriptstyle{j}i}\big\|_X\big),\ x\!\in\! \big[a,b\big]$  as in (3). We  $\gamma$  ; ;  $\psi$  $\rho$  ,  $\mu$  ,  $\omega$ ;  $\psi^{\psi}_{\mu_i,\omega_i,b-}f_{ji}\big(x\big)$ ,  $x\!\in\![a,b]$  as in (3). We assume here that  $0\!<\!f_{2i}(y)\!<\!\infty\,$  on  $\,[a,b]$ ,  $i\!=\!1,...,m.$ 

Here we consider the kernel

$$
k_i^-(x, y) := k_i(x, y) = \begin{cases} \psi'(y)(\psi(y) - \psi(x))^{\mu_i - 1} E_{\rho_i, \mu_i}^{\gamma_i} [\omega_i(\psi(y) - \psi(x))^{\rho_i}] & x \leq y < b, \\ 0, a < y < x, \end{cases} \tag{179}
$$

 $i = 1,...,m$ .

Choose weight  $u(x) \ge 0$ , so that

ed Fractional Inequalities  
\n
$$
\overline{\psi_i}(y) := f_{2i}(y) \int_a^y u(x) \frac{k_i^-(x, y)}{\left(e_{\rho_i, \mu_i, \omega_i, b}^{\gamma_i; \psi} - f_{2i}\right)(x)} dx < \infty,
$$
\n(180)  
\n
$$
\text{egrable on } [a, b], i = 1, \dots, m.
$$

a.e. on  $\big[a,b\big]$ , and that  $\psi_i$  is integrable on  $\big[a,b\big]$ ,  $i=1,...,m.$ 

Theorem 9 immediately implies:

**Theorem 58** All as in Remark 57. Let  $|p_i>1$  :  $\sum_{i=1}^{m}$   $\frac{1}{n}$   $=$   $1$  .  $=$ 1  $P_i$ m i  $p_i > 1$  :  $\sum_{i=1}^n \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex

and increasing. Then

y Type Generalized Fractional Inequalities  
\n
$$
\overline{\psi_i}(y) := f_{2i}(y) \int_a^y u(x) \frac{k_i^-(x, y)}{(e_{\rho_1, \mu_1, \omega_1, b}^{\gamma_1, \psi} f_{2i})(x)} dx < \infty,
$$
\n
$$
\text{that } \overline{\psi_i} \text{ is integrable on } [a, b], i = 1, ..., m.
$$
\n
$$
\text{ediately implies:}
$$
\nas in Remark 57. Let } p\_i > 1: \sum\_{i=1}^m \frac{1}{p\_i} = 1. \text{ Let the functions } \Phi\_i: \mathbb{R}\_+ \to \mathbb{R}\_+, i = 1, ..., m, \text{ be convex}\nan\n
$$
\int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \frac{\left| \left( e_{\rho_1, \mu_1, \omega_1, b}^{\gamma_1, \psi_1, \phi_1, b} f_1(x) \right) \right|}{\left( e_{\rho_1, \mu_1, \omega_1, b}^{\gamma_1, \psi_1, \phi_1, b} f_2(x) \right)} \right) dx \leq \prod_{i=1}^m \left( \int_a^b \overline{\psi_i}(y) \Phi_i \left( \frac{\left| f_{1i}(y) \right|}{f_{2i}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}}.
$$
\n
$$
\text{Let } \quad j = 1, 2; i = 1, ..., n; \quad \text{and } \quad \varphi_i > 0, \quad \varphi_i > 0, \quad \varphi_i > 0, \quad \text{and } \quad \varphi_i > 0, \quad \varphi_i > 0,
$$

We make

Remark 59 Let  $j=1,2;$   $i=1,...,n;$   $\mathop{\rho_i}\limits_{},\mathop{\mu_i}\limits_{},\omega_i>0,$   $\mathop{\gamma_i}\limits_{}<0,$   $N_i= \lceil \mathop{\mu_i}\rceil,$   $\mathop{\mu_i}\limits_{}\not\in \mathsf{N};$   $\theta:=\max\bigl(N_1,...,N_m\bigr),$  $\psi \in C^{\theta}([a,b])$ ,  $\psi'(x) \neq 0$  over  $[a,b]$ ,  $\psi$  is increasing;  $f_{ii} \in C^{N_i}([a,b])$  and  $f_{ii\psi}^{[N_i]}(x) = \left(\frac{1}{\psi'(x)}\right)^{N_i} f_{ii}(x)$ Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex<br>  $\prod_{a} \left( \int_a^b \overline{\psi_i}(y) \Phi_i \left( \frac{|f_{1i}(y)|}{f_{2i}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}}$ . (181)<br>  $0, \gamma_i < 0, N_i = [\mu_i], \qquad \mu_i \notin \mathbb{N}; \theta := \max(N_1,..., N_m),$ <br>  $\sum_{a} \in C^{N_i} ([a, b])$  and  $f_{j_i \psi}^{[$ dx d  $f_{ji\psi}^{[N_i]}(x) = \left(\frac{1}{\psi'(x)}\frac{a}{dx}\right) f_{ji}$  $N_i$  $\int_{ii\psi}^{N_i} f(x) dx = \left( \frac{1}{\psi(x)} \frac{d}{dx} \right)$ J <sup>1</sup>  $\overline{\phantom{a}}$  $\setminus$ ſ  $\int_{\psi}^{N_i} f(x) dx = \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right) f_{ji}(x), \forall i$  $x \in [a, b]$ , Here 57. Let  $p_i > 1: \sum_{i=1}^{m} \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i: \mathbb{R}_+ \to \mathbb{R}_+, i = 1,...,m$ , be convex<br>  $\sum_{i=1}^{r_i w_i} \frac{p_i}{p_i} = 1$ . Let the functions  $\Phi_i: \mathbb{R}_+ \to \mathbb{R}_+, i = 1,...,m$ , be convex<br>  $\sum_{i=1}^{r_i w_i} \frac{p_i}{p_i} = \prod_{i=1}^{m$  $\frac{1}{N_i}$  = 1. Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex<br>  $\leq \prod_{i=1}^{m} \left( \int_a^b \overline{\psi_i}(y) \Phi_i \left( \frac{|f_{1i}(y)|}{f_{2i}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}}$ . (181)<br>  $\mu_i, \omega_i > 0, \gamma_i < 0, N_i = [\mu_i], \quad \mu_i \notin \mathbb{N}; \theta := \max(N_1,...,N_m),$ <br>  $\$ 

$$
\left(CD_{\rho_i,\mu_i,\omega_i,a+}^{\gamma_i;\psi}f_{ji}\right)(x) \stackrel{(6)}{=} \left(e_{\rho_i,\nu_i-\mu_i,\omega_i,a+}^{-\gamma_i|\nu_i|}f_{ji\psi}\right)(x),\tag{182}
$$

 $\forall x \in [a,b]$ ,  $j = 1,2; i = 1,...,m$ .

We assume that  $0 \le f_{2i\psi}^{\lfloor N_i \rfloor}(y) \! < \! \infty \,$  on  $\big[a, b\big]$ ,  $i = 1, ... ,$  $\prod_{i\neq i}^{N_i} (y)$  <  $\infty$  on  $[a,b]$ ,  $i = 1,...,m$ .

Here we consider the kernel

$$
\left(\begin{array}{cc} C_{D_{\rho_1,\mu_1,\omega_1,\omega_1,\omega_1}}(x) \end{array}\right) = \left(\begin{array}{c} e^{-\gamma_1;\psi} \\ e^{-\gamma_1;\psi} \\ e^{-\gamma_1;\psi} \end{array}\right) \left(\chi\right),
$$
\n
$$
b, b, c \text{ and } c \in \mathbb{R} \text{ and
$$

 $i = 1,...,m$ .

Choose weight  $u \geq 0$ , so that

$$
{}^{C}\psi_{i}(y) := f_{2i\psi}^{[N_{i}]}(y) \int_{y}^{b} u(x) \frac{{}^{C}k_{i}^{+}(x,y)}{\left({}^{C}D_{\rho_{i},\mu_{i},\omega_{i},a+}^{y,w}f_{2i}\right)(x)}dx < \infty,
$$
\n(184)

a.e. on  $\big[a,b\big]$ , and that  $^{\,c}\psi_{i}$  is integrable on  $\big[a,b\big]$ ,  $i$  =  $1,...,m.$ 

Theorem 56 immediately produces:

**Theorem 60** All as in Remark 59. Let  $|p_i>1$  :  $\sum_{i=1}^{m} \frac{1}{n_i}=1$  .  $=$ 1  $P_i$ m i  $p_i > 1$  :  $\sum_{i=1}^n \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex and increasing. Then

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\nimmediately produces:

\n9 All as in Remark 59. Let 
$$
p_i > 1
$$
: 
$$
\sum_{i=1}^{m} \frac{1}{p_i} = 1
$$
. Let the functions  $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ ,  $i = 1,...,m$ , be convex

\nThen

\n
$$
\int_a^b u(x) \prod_{i=1}^m \Phi_i \left( \frac{\left| \left( \sum_{j \in \mathcal{N}_i : \psi_0, a_i + f_1 \right)}^{\gamma_i : \psi_0, a_i + f_1 \right)} \right|}{\left( \sum_{j \in \mathcal{N}_i : \psi_0, a_i + f_2 \right)}^{\gamma_i : \psi_0, a_i + f_2 \right)} dx \leq \prod_{i=1}^m \left( \int_a^b \psi_i(y) \Phi_i \left( \frac{\left| f_{1 \mid i \neq j}^{[N_i]}(y) \right|}{f_{2 \mid i \neq j}^{[N_i]}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}}.
$$
\n(185)

\n64. Let  $i = 1, 2, i = 1, \dots, n$ ,  $\psi_i \in \mathbb{R}$ , and  $\psi_i \in \mathbb{R}$ .

We make

**Remark 61** Let  $j=1,2;$   $i=1,...,n;$   $\rho_{_i}, \mu_{_i}, \omega_{_i}>0,$   $\gamma_{_i}< 0,$   $N_{_i} = \lceil \mu_{_i} \rceil,$   $\mu_{_i} \not\in \mathsf{N};$   $\theta := \max\bigl(N_1,...,N_m\bigr),$  $\psi \in C^{\theta}([a,b])$ ,  $\psi'(x) \neq 0$  over  $[a,b]$ ,  $\psi$  is increasing;  $f_{ii} \in C^{N_i}([a,b])$  and  $f_{ii\psi}^{[N_i]}(x) = \left(\frac{1}{\psi'(x)}\right)^{N_i} f_{ii}(x)$ journal of Advances in Applied & Computational Mathematics, 8, 2021<br>
Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex<br>
Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex<br>  $\left(\int_a^{b^C} \psi_i(y) \Phi_i \left(\frac{$ dx d  $f_{ji\psi}^{[N_i]}(x) = \left(\frac{1}{\psi^{'}(x)}\frac{a}{dx}\right) f_{ji}$  $N_i$  $\int_{ii\psi}^{N_i} f(x) dx = \left( \frac{1}{\psi(x)} \frac{d}{dx} \right)$ J ).  $\overline{ }$  $\setminus$ ſ  $\int_{\psi}^{N_i} f(x) dx = \left( \frac{1}{\psi'(x)} \frac{d}{dx} \right) f_{ji}(x), \forall i$  $x \in [a, b]$ , Here rk 59. Let  $p_i > 1: \sum_{i=1}^{m} \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ ,  $i = 1,...,m$ , be convex<br>  $D_{p_1, p_1, p_2, q_3}^{y_1, y_2, y_3, y_4} f_1(x)$ <br>  $D_{p_1, p_1, p_2, q_3}^{y_1, y_2, y_4} f_2(x)$ <br>  $\left| \int_a^b f_2^{(y)}(y) \Phi_i \left( \frac{\int_{0$  $\frac{1}{2^{n}} \sum_{i=1}^{n} \frac{1}{p_{i}} = 1.$  Let the functions  $\Phi_{i}: \mathsf{R}_{+} \to \mathsf{R}_{+}, i = 1,...,m$ , be convex<br>  $dx \leq \prod_{i=1}^{m} \left( \int_{a}^{b^{C}} \psi_{i}(y) \Phi_{i} \left( \frac{|f_{i\psi_{i}}^{[N_{i}]}(y)|}{f_{i\psi_{i}}^{[N_{i}]}(y)} \right)^{p_{i}} dy \right)^{\frac{1}{p_{i}}}.$  (185)<br>  $\phi_{i}, \mu_{i$ 

$$
\left(CD_{\rho_i,\mu_i,\omega_i,b}^{\gamma_i;\psi} - f_{ji}\right)(x) = (-1)^{N_i} \left(e_{\rho_i,N_i-\mu_i,\omega_i,b}^{-\gamma_i;\psi} - f_{ji\psi}^{[N_i]}\right)(x),\tag{186}
$$

 $\forall x \in [a,b]$ ,  $j = 1,2; i = 1,...,m$ .

We assume that  $0 \le f_{2i\psi}^{\lfloor N_i \rfloor}(y) \! < \! \infty \,$  on  $\big[a,b\big]$   $i = 1,...,n$  $\prod_{i\neq i}^{N_i} (y)$  <  $\infty$  on  $[a,b]$ ,  $i = 1,...,m$ .

Here we consider the kernel

$$
\left(\binom{C}{D_{\rho_1,\mu_1,\omega_1,b}^{x_i,\psi}}\binom{C}{\rho_1,\mu_1,\omega_1,b} \binom{C}{\rho_1,\mu_1,\omega_1,b} \binom{C}{\rho_1,\mu_1,\omega_1,b} \binom{C}{\rho_1,\mu_1,\omega_1,b} \binom{C}{\rho_1,\mu_1,b} \binom{C}{\rho_1
$$

 $i = 1,...,m$ .

Choose weight  $u \geq 0$ , so that

$$
{}^{C}\overline{\psi_{i}}(y):=f_{2i\psi}^{[N_{i}]}(y)\int_{a}^{y}u(x)\frac{{}^{C}k_{i}^{-}(x,y)}{\left({}^{C}D_{\rho_{i}^{*},\mu_{i},\omega_{i},b-}^{y,\psi}f_{2i}\right)(x)}dx<\infty,
$$
\n(188)

a.e. on  $\big[a,b\big]$ , and that  ${}^C\overline{\psi_{i}}$  is integrable on  $\big[a,b\big]$ ,  $i$  = 1,..., $m$ .

Theorem 58 immediately produces:

**Theorem 62** All as in Remark 61. Let  $|p_j>1$  :  $\sum_{i=1}^{m} \frac{1}{n_i}=1$  .  $=$ 1  $P_i$ m i  $p_i > 1$  :  $\sum_{i=1}^n \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex and increasing. Then

$$
\int_{a}^{b} u(x) \prod_{i=1}^{m} \Phi_{i} \left( \frac{\left| \left( {}^{C}D_{\rho_{i}, \mu_{i}, \omega_{i}, b}^{y_{i}, w_{i}, b-f_{1i}}(x) \right) \right|}{\left( {}^{C}D_{\rho_{i}, \mu_{i}, \omega_{i}, b-f_{2i}}^{y_{i}, w_{i}, b-f_{2i}}(x) \right)} \right) dx \leq \prod_{i=1}^{m} \left( \int_{a}^{b^{C}} \overline{\psi_{i}}(y) \Phi_{i} \left( \frac{\left| f_{1i\psi}^{[N_{i}]}(y) \right|}{f_{2i\psi}^{[N_{i}]}(y)} \right|^{p_{i}} dy \right)^{\frac{1}{p_{i}}}.
$$
\n(189)

We make

**Remark 63** Let  $j=1,2; i=1,...,m;$   $\rho_{_i}, \mu_{_i}, \omega_{_i} > 0,$   $\gamma_{_i} < 0,$   $N_{_i} = \lceil \mu_{_i} \rceil,$   $\mu_{_i} \notin \mathsf{N};$   $\theta := \max\bigl(N_1,...,N_m\bigr),$  $\psi\in C^{\theta}([a,b]),$   $\psi^{'}(x) \neq 0$  over  $[a,b],$   $\psi$  is increasing;  $f_{ji}\in C([a,b]).$  Let  $0\leq \beta_i\leq 1$  and  $\xi_i=\mu_i+\beta_i(N_i-\mu_i)$ ,  $i=1,...,m$  . We assume that  $^{RL}D^{\gamma_{i}^{'[1-\beta_{i}\},\psi}_{\rho_{i},\xi_{i},\omega_{i},a}+f_{ji}\in C([a,b])$  and  $0<^{RL}D^{\gamma_{i}^{'[1-\beta_{i}\},\psi}_{\rho_{i},\xi_{i},\omega_{i},a}+f_{ji}\in C([a,b])$  $^{RL}D^{\gamma_i(l-\beta_i)w}_{\rho_i,\xi_i,\omega_i,a+}f_{ji} \in C([a,$  $^{+}$  $\gamma$ ,  $(1-\beta)$ ,  $\mu$  $\gamma_i^{(1-\beta_i)k\mu}_{\rho_i,\xi_i,\omega_i,a+} f_{ji} \in C([a,b])$  and  $0 <^{RL}D^{\gamma_i^{(1-\beta_i)k\mu}}_{\rho_i,\xi_i,\omega_i,a+} f_{2i}(y) < \infty$  on  $[a,b]$ ,  $i=1,...$ RL  $\sum_{i}$   $\gamma_i$   $(1-\beta_i)$ ;  $\psi$  $\mathbb{P}^{(1|\mathcal{A}_i), \mathbb{W}}_{\rho_i, \xi_i, \omega_i, a+} f_{2i}(y) \! < \infty$  on  $[a, b]$ ,  $i = 1, ..., m$  . Here we have alized Fractional Inequalities<br>  $\left(\left|\left(\frac{D_{\rho_1,\mu_1,\omega_1,b}^{Y,\Psi}-f_{11}}{D_{\rho_1,\mu_1,\omega_1,b}+f_{11}}\right)\right|dx \leq \prod_{i=1}^{m} \left(\int_a^{b^C} \overline{\psi_i}(y) \Phi_i\left(\frac{\left|f_{1|\psi_i}^{[V_i]}(y)\right|}{f_{2|\psi}^{[V_i]}(y)}\right|^p dy\right)^{\frac{1}{p_i}}$ . (189)<br>  $j=1,2; i=1,...,m; \rho_1,\mu_1,\omega$ ies<br>  $\begin{align*}\n\left(\frac{\partial}{\partial t}\right) \left|dx \leq \prod_{i=1}^{m} \left( \int_{a}^{b^{C}} \overline{\psi_{i}}(y) \Phi_{i}\left(\frac{\int_{x_{i}^{k_{i}}}^{x_{i}^{j}}(y)}{\int_{x_{i}^{k_{i}}}^{x_{i}}(y)}\right)^{p_{i}} dy \right)^{\frac{1}{p_{i}}}, \qquad (189) \\
\frac{\partial}{\partial t} \mathcal{L} &\leq \prod_{i=1}^{m} \left( \int_{a}^{b^{C}} \overline{\psi_{i}}(y) \Phi_{i}\left(\frac{\int_{x_{i}^{k_{i$ 

$$
\left(H\mathbf{D}_{\rho_{i},\mu_{i},\omega_{i},a+}^{\gamma_{i},\beta_{i};\psi}f_{j}x\right) = e_{\rho_{i},\xi_{i}-\mu_{i},\omega_{i},a+}^{\gamma_{i}\beta_{i};\psi} D_{\rho_{i},\xi_{i},\omega_{i},a+}^{\gamma_{i}\left(1-\beta_{i}\right)\psi}f_{j}x\right),\tag{190}
$$

 $\forall x \in [a,b]$ ,  $j = 1,2; i = 1,...,m$ .

Here we consider the kernel

The assume that 
$$
\sum_{p_1, \xi_1, \omega_1, a_1, j} \sum_{(w_1, w_1, a_1, j)} \sum_{(w_1, w_1, w_1, a_1, j)} \sum_{(w_1, w_1, w_1, a_1, j)} \sum_{(w_1, w_1, w_1, a_1, a_1, j)} \sum_{(p_1, \xi_1, w_1, a_1, a_1, j)} \sum_{(p_1, \xi_1, w_1, a_1, a_1, j)} \sum_{(p_1, \xi_1, w_1, a_1, a_1, j)} \sum_{(p_1, \xi_1, w_1, a_1, j)} \sum_{(p_1, \xi_1, w_1
$$

 $i = 1,...,m$ .

Choose weight  $u \geq 0$ , so that

$$
{}^{P}\psi_{i}(y) := \left({}^{RL}D_{\rho_{i},\xi_{i},\omega_{i},a+}^{\gamma_{i}(1-\beta_{i})\psi}f_{2i}(y)\right) \int_{y}^{b} \frac{u(x)^{P}k_{i}^{+}(x,y)}{\left({}^{H}D_{\rho_{i},\mu_{i},\omega_{i},a+}^{\gamma_{i},\beta_{i}\psi}f_{2i}\right)(x)}dx < \infty,
$$
\n(192)

a.e. on  $\big[a,b\big]$ , and that  $^{\,p}\mathbb{W}_{i}$  is integrable on  $\big[a,b\big]$ ,  $i$  =  $1,...,m.$ 

Theorem 56 immediately produces:

**Theorem 64** All as in Remark 63. Let  $|p_j>1$  :  $\sum_{i=1}^{m} \frac{1}{n_i}=1$  .  $=$ 1  $P_i$ m i  $p_i > 1$  :  $\sum_{i=1}^n \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex and increasing. Then

$$
[0, x < y < b,
$$
\n
$$
m.
$$
\n\nwe weight  $u \ge 0$ , so that

\n
$$
{}^{p}\psi_{i}(y) := \left(\binom{k}{2} \sum_{j_{i},j_{i},\omega_{j},a+}^{y_{i}(\lfloor n\beta_{i}\rfloor)\psi} f_{j_{i}} \left(\frac{u(x)^{p} k_{i}^{+}(x, y)}{p_{j_{i},\omega_{i},\omega_{i},a+} f_{2i}(x)} dx < \infty, \right)
$$
\n
$$
[192]
$$
\nand that  ${}^{p}\psi_{i}$  is integrable on  $[a,b]$ ,  $i = 1,...,m$ .

\nfrom 56 immediately produces:

\nfrom 64 All as in Remark 63. Let  $p_{i} > 1: \sum_{i=1}^{m} \frac{1}{p_{i}} = 1$ . Let the functions  $\Phi_{i}: R_{+} \to R_{+}, i = 1,...,m$ , be convex as a single. Then

\n
$$
\int_{a}^{b} u(x) \prod_{i=1}^{m} \Phi_{i} \left( \frac{\left|\binom{H}{i} D_{j_{i},\omega_{i},a+}^{y_{i},\omega_{i},a+} f_{1i}(x)\right|}{\binom{H}{i} D_{j_{i},\omega_{i},\omega_{i},a+}^{y_{i},\omega_{i},a+} f_{2i}(x)} \right) dx \leq \prod_{i=1}^{m} \left( \int_{a}^{b} \psi_{i}(y) \Phi_{i} \left( \frac{\left|\binom{k}{i} D_{j_{i},\omega_{i},\omega_{i},a+}^{y_{i},\omega_{i},a+} f_{1i}(y)\right|}{\binom{k}{i} D_{j_{i},\omega_{i},\omega_{i},a+}^{y_{i},\omega_{i},\omega_{i},a+} f_{2i}(y)} \right|^{p_{i}} dy \right)^{\frac{1}{p_{i}}}.
$$
\nand the

We make

Remark 65 Let  $j=1,2;$   $i=1,...,m;$   $\mathop{\rho_i}\limits_{},\mathop{\mu_i}\limits_{},\mathop{\omega_i}\limits_{}\!>0,$   $\mathop{\gamma_i}\limits_{}<0,$   $N_i$   $= \lceil\mathop{\mu_i}\rceil,$   $\mathop{\mu_i}\limits_{}\notin \mathsf{N};$   $\mathop{\theta}:=\max\bigl(N_1,...,N_m\bigr),$  $\psi \in C^{\theta}([a,b])$ ,  $\psi'(x) \neq 0$  over  $[a,b]$ ,  $\psi$  is increasing;  $f_{ji} \in C([a,b])$ . Let  $0 \leq \beta_i \leq 1$  and  $\xi_i = \mu_i + \beta_i (N_i - \mu_i)$ ,  $i=1,...,m$  . We assume that  $^{RL}D^{\gamma_{i}[1-\beta_{i}],\Downarrow}_{\rho_{i},\xi_{i},\omega_{i},b}-f_{ji}\in C([a,b])$  and  $0<^{RL}D^{\gamma_{i}[1-\beta_{i}],\Downarrow}_{\rho_{i},\xi_{i},\omega_{i},b}-f_{ji}$  $\int_{\rho_i,\xi_i,\omega_i,b-}^{\gamma_i(1-\beta_i),\psi} f_{ji} \in C([a,$ Ξ,  $\gamma$ ,  $(1-\beta)$ ,  $\psi$  $\sum_{\substack{p_i,\xi_i,\omega_i,b-f_{ji} \ \in C([a,b]) \text{ and } 0 <^{RL} \ D^{\gamma_i [1-\beta_i] \psi}_{\rho_i,\xi_i,\omega_i,b}-f_{2i}(y) < \infty \text{ on } [a,b], i=1,...}$ RL  $\sum_{i}$   $\gamma_i$   $(1-\beta_i)$ ;  $\psi$  $\mathbb{P}^{(1|\text{-}\textcolor{black}{\beta}_i|)\textcolor{black}{w}}_{\rho_i,\xi_i,\omega_i,b-\textcolor{black}{f}_{2i}}(y)\!<\!\infty$  on  $[a,b]$  ,  $i\!=\!1,...,m$  . Here we have Journal of Advances in Applied & Computational Mathematics, 8, 2021<br>  $j = 1, 2; i = 1, ..., m; \rho_1, \mu_1, \omega_i > 0, \gamma_i < 0, N_i = \lceil \mu_i \rceil, \qquad \mu_i \notin \mathbb{N}; \theta := \max(N_1, ..., N_m),$ <br>  $\text{Pr}\left[a, b\right]$ ,  $\psi$  is increasing;  $f_{ji} \in C([a, b])$ . Let  $0 \le \beta_i \le 1$ Journal of Advances in Applied & Computational Mathematics, 8, 2021<br>  $\rho_i, \mu_i, \omega_i > 0, \gamma_i < 0, N_i = \lceil \mu_i \rceil, \qquad \mu_i \notin \mathbb{N}; \theta := \max(N_1, ..., N_m),$ <br>
aasing;  $f_{ji} \in C([a, b])$ . Let  $0 \le \beta_i \le 1$  and  $\xi_i = \mu_i + \beta_i (N_i - \mu_i),$ <br>  $[a, b]$  and  $0 < \sum_{i$ 

1 ; , , , ; , , , (16) , ; , , , f x e D f x ji i i b i i i RL i i b i i i i ji i i b i i i H <sup>D</sup> (194)

 $\forall x \in [a,b]$ ,  $j = 1,2; i = 1,...,m$ .

Here we consider the kernel

$$
[H] \text{ (where given in the image)} \quad \text{where } \quad \text
$$

 $i = 1, \ldots, m$ .

Choose weight  $u \geq 0$ , so that

$$
{}^{P}\overline{\psi_{i}}(y) := \left( {}^{RL}D_{\rho_{i},\xi_{i},\omega_{i},b}^{\gamma_{i}(1-\beta_{i})\psi} f_{2i}(y) \right) \int_{a}^{y} \frac{u(x)^{P} k_{i}^{-}(x,y)}{\left( {}^{H}D_{\rho_{i},\mu_{i},\omega_{i},b}^{\gamma_{i},\beta_{i},\psi} - f_{2i}(x) \right)} dx < \infty, \tag{196}
$$

a.e. on  $\big[a,b\big]$ , and that  $^P\overline{\psi_{i}}$  is integrable on  $\big[a,b\big]$ ,  $i=1,...,m.$ 

Theorem 58 immediately produces:

**Theorem 66** All as in Remark 65. Let  $|p_i>1$  :  $\sum_{i=1}^{m} \frac{1}{n_i}=1$  .  $=$ 1  $P_i$ m i  $p_i > 1$  :  $\sum_{i=1}^n \frac{1}{p_i} = 1$ . Let the functions  $\Phi_i : \mathsf{R}_+ \to \mathsf{R}_+$ ,  $i = 1,...,m$ , be convex

and increasing. Then

$$
[0, a < y < x,
$$
\n*m.*\n\n
$$
\text{se weight } u \ge 0, \text{ so that}
$$
\n
$$
{}^{p} \overline{\psi_{i}}(y) := \left(\binom{k}{p_{i}, \xi_{i}, \omega_{i}, b} f_{2i}(y)\right) \int_{a}^{y} \frac{u(x)^{p} k_{i}(x, y)}{(H D_{\rho_{i}, \psi_{i}, \omega_{i}, b} f_{2i}(x))} dx < \infty,
$$
\n(196)\n\n*a, b*, and that\n
$$
{}^{p} \overline{\psi_{i}} \text{ is integrable on } [a, b], i = 1, \dots, m.
$$
\n*n.e.*\n\n*n*\n\n
$$
\text{se immediately produces:}
$$
\n
$$
\text{rem 66 All as in Remark 65. Let } p_{i} > 1: \sum_{i=1}^{m} \frac{1}{p_{i}} = 1. \text{ Let the functions } \Phi_{i}: R_{+} \to R_{+}, i = 1, \dots, m, \text{ be convex}
$$
\n\n
$$
\text{asing. Then}
$$
\n
$$
\int_{a}^{b} u(x) \prod_{i=1}^{m} \Phi_{i} \left( \frac{\left(\left(\frac{H}{D_{\rho_{i}, \psi_{i}, \omega_{i}, b} f_{1i}(x)}\right)}{\left(\frac{H}{D_{\rho_{i}, \psi_{i}, \omega_{i}, b} f_{2i}(x)}\right)}\right) dx \leq \prod_{i=1}^{m} \left(\frac{\binom{k}{2} \frac{L}{\psi_{i}} \left(\frac{L}{\psi_{i}} \frac{L}{\psi_{i}} \left(\frac{L}{\psi_{i}} \frac{L}{\psi_{i}} \frac{L}{\psi_{
$$

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