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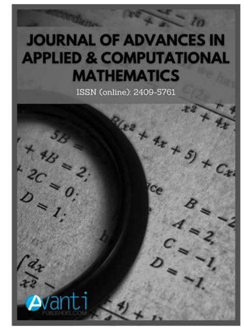

Vectorial Prabhakar Hardy Type Generalized Fractional Inequalities under Convexity

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ABSTRACT

We present a detailed great variety of Hardy type fractional inequalities under convexity and *Lp* norm in the setting of generalized Prabhakar and Hilfer fractional calculi of left and right integrals and derivatives. The radial multivariate case of the above over a spherical shell is developed in detail to all directions. Many inequalities are of vectorial splitting rational *Lp* type or of separating rational *Lp* type, others involve ratios of functions and of fractional integral operators.

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1. Background

This work is inspired by [3-11].

Here we consider the Prabhakar function (also known as the three parameter Mittag-Laffler function), (see [6], p. 97; [5])

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{k!\Gamma(\alpha k+\beta)}z^k, \tag{1}$$

where Γ is the gamma function; $\alpha,\beta,\gamma\in\mathbb{R}:\alpha,\beta>0,\ z\in\mathbb{R},$ and $(\gamma)_k=\gamma(\gamma+1)\ldots(\gamma+k-1)$. It is $E_{\alpha,\beta}^{0}(z)=\frac{1}{\Gamma(\beta)}$.

Here we follow [4].

Let $a,b\in\mathbb{R}$, $a<b$ and $x\in[a,b]$; $f\in C([a,b])$. Let also $\psi\in C^1([a,b])$ which is increasing. The left and right Prabhakar fractional integrals with respect to ψ are defined as follows:

$$\left(e_{\rho,\mu,\omega,a+}^{\gamma;\psi}f\right)(x)=\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\mu-1}E_{\rho,\mu}^{\gamma}\left[\omega(\psi(x)-\psi(t))^{\rho}\right]f(t)dt, \tag{2}$$

and

$$\left(e_{\rho,\mu,\omega,b-}^{\gamma;\psi}f\right)(x)=\int_x^b\psi'(t)(\psi(t)-\psi(x))^{\mu-1}E_{\rho,\mu}^{\gamma}\left[\omega(\psi(t)-\psi(x))^{\rho}\right]f(t)dt, \tag{3}$$

where $\rho,\mu>0;\ \gamma,\omega\in\mathbb{R}$.

Functions (2) and (3) are continuous ([4]).

Next, additionally, assume that $\psi'(x)\neq 0$ over $[a,b]$ and let $\psi,f\in C^N([a,b])$, where $N=\lceil\mu\rceil$, ($\lceil\cdot\rceil$ is the ceiling of the number), $0<\mu\notin\mathbb{N}$. We define the ψ-Prabhakar-Caputo left and right fractional derivatives of order μ ([4]) as follows ($x\in[a,b]$):

$$\left({}^{C}D_{\rho,\mu,\omega,a+}^{\gamma;\psi}f\right)(x)=\int_a^x\psi'(t)(\psi(x)-\psi(t))^{N-\mu-1}\quad E_{\rho,N-\mu}^{-\gamma}\left[\omega(\psi(x)-\psi(t))^{\rho}\right]\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^N f(t)dt, \tag{4}$$

and

$$\left({}^{C}D_{\rho,\mu,\omega,b-}^{\gamma;\psi}f\right)(x)=(-1)^N\int_x^b\psi'(t)(\psi(t)-\psi(x))^{N-\mu-1}\quad E_{\rho,N-\mu}^{-\gamma}\left[\omega(\psi(t)-\psi(x))^{\rho}\right]\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^N f(t)dt. \tag{5}$$

One can write these (see (4), (5)) as

$$\left({}^{C}D_{\rho,\mu,\omega,a+}^{\gamma;\psi}f\right)(x)=\left(e_{\rho,N-\mu,\omega,a+}^{-\gamma;\psi}f_{\psi}^{[N]}\right)(x), \tag{6}$$

and

$$\left({}^{C}D^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x)=(-1)^{N}\left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,b-}f^{[N]}_{\psi}\right)(x), \tag{7}$$

where

$$f^{[N]}_{\psi}(x)=f^{(N)}_{\psi}f(x):=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}f(x), \tag{8}$$

$\forall\ x\in[a,b]$.

Functions (6) and (7) are continuous on $[a,b]$.

Next we define the ψ-Prabhakar-Riemann Liouville left and right fractional derivatives of order μ ([4]) as follows ($x\in[a,b]$, $f\in C([a,b])$):

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\int_{a}^{x}\psi'(t)(\psi(x)-\psi(t))^{N-\mu-1}\quad E^{-\gamma}_{\rho,N-\mu}\left[\omega(\psi(x)-\psi(t))^{\rho}\right]f(t)dt, \tag{9}$$

and

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x)=\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\int_{x}^{b}\psi'(t)(\psi(t)-\psi(x))^{N-\mu-1}\quad E^{-\gamma}_{\rho,N-\mu}\left[\omega(\psi(t)-\psi(x))^{\rho}\right]f(t)dt. \tag{10}$$

That is we have

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,a+}f\right)(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,a+}f\right)(x), \tag{11}$$

and

$$\left({}^{RL}D^{\gamma;\psi}_{\rho,\mu,\omega,b-}f\right)(x)=\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}\left(e^{-\gamma;\psi}_{\rho,N-\mu,\omega,b-}f\right)(x), \tag{12}$$

$\forall\ x\in[a,b]$.

We define also the ψ-Hilfer-Prabhakar left and right fractional derivatives of order μ and type $0\leq\beta\leq1$ ([4]), as follows

$$\left({}^{H}\mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,a+}f\right)(x)=e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,a+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,a+}f(x), \tag{13}$$

and

$$\left({}^{H}\mathsf{D}^{\gamma,\beta;\psi}_{\rho,\mu,\omega,b-}f\right)(x)=e^{-\gamma\beta;\psi}_{\rho,\beta(N-\mu),\omega,b-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N}e^{-\gamma(1-\beta);\psi}_{\rho,(1-\beta)(N-\mu),\omega,b-}f(x), \tag{14}$$

$\forall\ x\in[a,b]$.

When $\beta = 0$, we get the Riemann-Liouville version, and when $\beta = 1$, we get the Caputo version.

We call $\xi = \mu + \beta(N-\mu)$, we have that $N-1 < \mu \leq \mu + \beta(N-\mu) \leq \mu + N - \mu = N$, hence $\lceil \xi \rceil = N$.

We can easily write that

$$\left({}^{H}\mathsf{D}_{\rho,\mu,\omega,a+}^{\gamma,\beta;\psi} f\right)(x) = e_{\rho,\xi-\mu,\omega,a+}^{-\gamma\beta;\psi} \, {}^{RL}D_{\rho,\xi,\omega,a+}^{\gamma(1-\beta);\psi} f(x), \tag{15}$$

and

$$\left({}^{H}\mathsf{D}_{\rho,\mu,\omega,b-}^{\gamma,\beta;\psi} f\right)(x) = e_{\rho,\xi-\mu,\omega,b-}^{-\gamma\beta;\psi} \, {}^{RL}D_{\rho,\xi,\omega,b-}^{\gamma(1-\beta);\psi} f(x), \tag{16}$$

$\forall \; x \in [a,b]$.

In this work we develop a great variety of fractional inequalities of Hardy type invovling convexity and engaging the above exposed: ψ-Prabhakar fractional left and right fractional integrals, the ψ-Prabhakar-Caputo left and right fractional derivatives, the ψ-Riemann-Liouville left and right fractional derivatives, and the ψ-Hilfer-Prabhakar left and right fractional derivatives. The radial multivariate case of all of the above over a spherical shell is studied in full detail. We involve ratios of functions and of integral operators and we produce among others vectorial splitting rational L_p inequalities, as well as separating rational L_p inequalities.

2. Prerequisites

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k(x,\cdot)$ measurable on Ω_2, and

$$K(x) = \int_{\Omega_2} k(x,y) d\mu_2(y), \text{ for any } x \in \Omega_1. \tag{17}$$

We suppose that $K(x) > 0$ a.e. on Ω_1 and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g_i : \Omega_1 \to \mathbb{R}$, $i = 1,\ldots,n,$ with the representation

$$g_i(x) = \int_{\Omega_2} k(x,y) f_i(y) d\mu_2(y), \tag{18}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1,\ldots,n$.

Denote by $\vec{x} = x := (x_1,\ldots,x_n) \in \mathbb{R}^n$, $\vec{g} := (g_1,\ldots,g_n)$ and $\vec{f} := (f_1,\ldots,f_n)$.

We consider here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ a convex function, which is increasing per coordinate, i.e. if $x_i \leq y_i$, $i = 1,\ldots,n$, then

$$\Phi(x_1,\ldots,x_n) \leq \Phi(y_1,\ldots,y_n).$$

In [3], p. 588, we proved that

Theorem 1 Let u be a weight function on Ω_1, and k, K, g_i, f_i, $i = 1,\ldots,n \in \mathbb{N}$, and Φ defined as above. Assume that the function $x \to u(x)\dfrac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v on Ω_2 by

$$v(y) := \int_{\Omega_1} u(x) \frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{19}$$

Then

$$\int_{\Omega_1} u(x) \Phi\left(\frac{|g_1(x)|}{K(x)}, ..., \frac{|g_n(x)|}{K(x)} \right) d\mu_1(x) \le \int_{\Omega_2} v(y) \Phi(|f_1(y)|, ..., |f_n(y)|) d\mu_2(y), \tag{20}$$

under the assumptions:

(i) f_i, $\Phi(|f_1|, ..., |f_n|)$, are $k(x,y)d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$, for all $i = 1, ..., n$,

(ii) $v(y)\Phi(|f_1(y)|, ..., |f_n(y)|)$ is μ_2 -integrable.

Notation 2 From now on we may write

$$\vec{g}(x) = \int_{\Omega_2} k(x,y) \vec{f}(y) d\mu_2(y), \tag{21}$$

which means

$$(g_1(x), ..., g_n(x)) = \left(\int_{\Omega_2} k(x,y) f_1(y) d\mu_2(y), ..., \int_{\Omega_2} k(x,y) f_n(y) d\mu_2(y) \right). \tag{22}$$

Similarly, we may write

$$|\vec{g}(x)| = \left| \int_{\Omega_2} k(x,y) \vec{f}(y) d\mu_2(y) \right|, \tag{23}$$

and we mean

$$(|g_1(x)|, ..., |g_n(x)|) = \left(\left| \int_{\Omega_2} k(x,y) f_1(y) d\mu_2(y) \right|, ..., \left| \int_{\Omega_2} k(x,y) f_n(y) d\mu_2(y) \right| \right). \tag{24}$$

We also can write that

$$|\vec{g}(x)| \le \int_{\Omega_2} k(x,y) \left| \vec{f}(y) \right| d\mu_2(y), \tag{25}$$

and we mean the fact that

$$|g_i(x)| \le \int_{\Omega_2} k(x,y) |f_i(y)| d\mu_2(y), \tag{26}$$

for all $i = 1, ..., n,$ etc.

Notation 3 Next let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ -finite measures, and let $k_j : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a nonnegative measurable function, $k_j(x, \cdot)$ measurable on Ω_2 and

$$K_j(x) = \int_{\Omega_2} k_j(x,y) d\mu_2(y), \ \ x \in \Omega_1, j = 1, ..., m. \tag{27}$$

We suppose that $K_j(x)>0$ a.e. on Ω_1. Let the measurable functions $g_{ji}:\Omega_1\to\mathbb{R}$ with the representation

$$g_{ji}(x)=\int_{\Omega_2}k_j(x,y)f_{ji}(y)d\mu_2(y), \tag{28}$$

where $f_{ji}:\Omega_2\to\mathbb{R}$ are measurable functions, $i=1,\ldots,n$ and $j=1,\ldots,m.$

Denote the function vectors $\vec{g_j}:=(g_{j1},g_{j2},\ldots,g_{jn})$ and $\vec{f_j}:=(f_{j1},\ldots,f_{jn}),\ j=1,\ldots,m.$

We say $\vec{f_j}$ is integrable with respect to measure μ, iff all f_{ji} are integrable with respect to μ.

We also consider here $\Phi_j:\mathbb{R}_+^n\to\mathbb{R}_+$, $j=1,\ldots,m,$ convex functions that are increasing per coordinate. Again u is a weight function on Ω_1.

We make

Remark 4 Following Notation 3, let $F_j:\Omega_2\to\mathbb{R}\cup\{\pm\infty\}$ be measurable functions, $j=1,\ldots,m,$ with $0<F_j(y)<\infty$ on Ω_2. In (27) we replace $k_j(x,y)$ by $k_j(x,y)F_j(y)$, $j=1,\ldots,m,$ and we have the modified $K_j(x)$ as

$$L_j(x):=\int_{\Omega_2}k_j(x,y)F_j(y)d\mu_2(y),\ x\in\Omega_1. \tag{29}$$

We assume $L_j(x)>0$ a.e. on Ω_1.

As new $\vec{f_j}$ we consider now $\vec{\gamma_j}:=\frac{\vec{f_j}}{F_j}$, $j=1,\ldots,m,$ where $\vec{f_j}=(f_{j1},\ldots,f_{jn})$; $\vec{\gamma_j}=\left(\frac{f_{j1}}{F_j},\ldots,\frac{f_{jn}}{F_j}\right).$

Notice that

$$g_{ji}(x)=\int_{\Omega_2}k_j(x,y)f_{ji}(y)d\mu_2(y)\quad=\int_{\Omega_2}\left(k_j(x,y)F_j(y)\right)\left(\frac{f_{ji}(y)}{F_j(y)}\right)d\mu_2(y), \tag{30}$$

$x\in\Omega_1$, all $j=1,\ldots,m;\ i=1,\ldots,n.$

So we can write

$$\vec{g_j}(x)=\int_{\Omega_2}\left(k_j(x,y)F_j(y)\right)\vec{\gamma_j}(y)d\mu_2(y),\ \ j=1,\ldots,m. \tag{31}$$

We mention

Theorem 5 ([3], p. 481) Here we follow Remark 4. Let $\rho\in\{1,\ldots,m\}$ be fixed. Assume that the function

$$x \mapsto \left(\frac{u(x)\left(\prod_{j=1}^{m} F_j(y)\right)\left(\prod_{j=1}^{m} k_j(x,y)\right)}{\prod_{j=1}^{m} L_j(x)} \right)$$

is integrable on Ω_1, for each $y \in \Omega_2$. Define U_m on Ω_2 by

$$U_m(y) := \left(\prod_{j=1}^{m} F_j(y)\right) \int_{\Omega_1} \frac{u(x)\prod_{j=1}^{m} k_j(x,y)}{\prod_{j=1}^{m} L_j(x)} d\mu_1(x) < \infty. \tag{32}$$

Then

$$\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j\left(\left|\frac{\overrightarrow{g_j}(x)}{L_j(x)}\right|\right) d\mu_1(x) \leq \left(\prod_{\substack{j=1 \\ j\neq\rho}}^{m} \int_{\Omega_2} \Phi_j\left(\left|\frac{\overrightarrow{f_j}(y)}{F_j(y)}\right|\right) d\mu_2(y)\right) \cdot \left(\int_{\Omega_2} \Phi_\rho\left(\left|\frac{\overrightarrow{f_\rho}(y)}{F_\rho(y)}\right|\right) U_m(y) d\mu_2(y)\right), \tag{33}$$

under the assumptions:

(i) $\frac{\overrightarrow{f_j}}{F_j}$, $\Phi_j\left(\frac{\left|\overrightarrow{f_j}\right|}{F_j}\right)$ are both $k_j(x,y)F_j(y)d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$, $j = 1,\ldots,m$,

(ii) $U_m \Phi_\rho\left(\frac{\overrightarrow{f_\rho}}{F_\rho}\right)$; $\Phi_1\left(\frac{\left|\overrightarrow{f_1}\right|}{F_1}\right), \Phi_2\left(\frac{\left|\overrightarrow{f_2}\right|}{F_2}\right), \ldots, \Phi_\rho\left(\frac{\left|\overrightarrow{f_\rho}\right|}{F_\rho}\right), \ldots, \Phi_m\left(\frac{\left|\overrightarrow{f_m}\right|}{F_m}\right)$, are μ_2 -integrable, where $\Phi_\rho\left(\frac{\left|\overrightarrow{f_\rho}\right|}{F_\rho}\right)$ is absent.

We also mention

Theorem 6 ([3], p. 519) Here all as in Notation 3 and Remark 4. Assume that the functions ($j = 1,2,\ldots,m \in \mathbb{N}$)

$$x \mapsto \left(\frac{u(x)k_j(x,y)F_j(y)}{K_j(x)} \right)$$

are integrable on Ω_1, for each fixed $y \in \Omega_2$. Define W_j on Ω_2 by

$$W_j(y) := \left(\int_{\Omega_1} \frac{u(x)k_j(x,y)}{K_j(x)} d\mu_1(x)\right) F_j(y) < \infty, \tag{34}$$

on Ω_2.

Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,\dots,m,$ be convex and increasing per coordinate.

Then

$$\int_{\Omega_1} u(x) \prod_{j=1}^{m} \Phi_j \left(\left| \frac{\overrightarrow{g_j}(x)}{L_j(x)} \right| \right) d\mu_1(x) \leq \prod_{j=1}^{m} \left(\int_{\Omega_2} W_j(y) \Phi_j \left(\left| \frac{\overrightarrow{f_j}(y)}{F_j(y)} \right| \right)^{p_j} d\mu_2(y) \right)^{\frac{1}{p_j}}, \tag{35}$$

under the assumptions:

(i) $\frac{\overrightarrow{f_j}}{F_j}, \Phi_j \left(\frac{\left|\overrightarrow{f_j}\right|}{F_j} \right)^{p_j}$ are both $k_j(x,y) F_j(y) d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$, $j = 1,\dots,m,$

(ii) $W_j \Phi_j \left(\frac{\left|\overrightarrow{f_j}\right|}{F_j} \right)^{p_j}$ is μ_2 -integrable, $j = 1,\dots,m.$

We make

Remark 7 Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k(x,\cdot)$ measurable on Ω_2, and

$$K(x) = \int_{\Omega_2} k(x,y) d\mu_2(y), \text{ for any} x \in \Omega_1.$$

We assume $K(x) > 0$ a.e. on Ω_1 and the weight functions are nonnegative functions on the related set. We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$, with the representation

$$g_i(x) = \int_{\Omega_2} k(x,y) f_i(y) d\mu_2(y),$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1,\dots,n$. Here u stands for a weight function on Ω_1. So we follow Notation 3 for $j = m = 1$. We write here $\vec{g} := (g_1,\dots,g_n)$, $\vec{f} := (f_1,\dots,f_n)$.

We set

$$\left\|\vec{f}(y)\right\|_\infty := \max\{|f_1(y)|,\dots,|f_n(y)|\},$$
and
$$\left\|\vec{f}(y)\right\|_q := \left(\sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, q \geq 1. \tag{36}$$

We assume that

$$0 < \left\| \overrightarrow{f}(y) \right\|_q < \infty, \text{ a.e.on} (a,b), \tag{37}$$

$1 \le q \le \infty$ fixed.

Let

$$L_q(x) := \int_{\Omega_2} k(x,y) \left\| \overrightarrow{f}(y) \right\|_q d\mu_2(y), \; x \in \Omega_1, \tag{38}$$

$1 \le q \le \infty$ fixed.

We assume $L_q(x) > 0$ a.e. on Ω_1.

We furher assume that the function

$$x \mapsto \left(\frac{u(x) k(x,y) \left\| \overrightarrow{f}(y) \right\|_q}{L_q(x)} \right) \tag{39}$$

is integrable on Ω_1, for almost each fixed $y \in \Omega_2$.

Define W_q on Ω_2 by

$$W_q(y) := \left(\int_{\Omega_1} \frac{u(x) k(x,y)}{L_q(x)} d\mu_1(x) \right) \left\| \overrightarrow{f}(y) \right\|_q < \infty, \tag{40}$$

a.e. on Ω_2.

Let

$$\overrightarrow{\gamma} := \left(\frac{f_1}{\left\| \overrightarrow{f}(y) \right\|_q}, \frac{f_2}{\left\| \overrightarrow{f}(y) \right\|_q}, \ldots, \frac{f_n}{\left\| \overrightarrow{f}(y) \right\|_q} \right), \tag{41}$$

i.e. $\overrightarrow{\gamma} = \dfrac{\overrightarrow{f}}{\left\| \overrightarrow{f}(y) \right\|_q}$.

Here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is a convex and increasing per coordinate function.

We mention

Theorem 8 ([3], p. 536) Let all here as in Remark 7. Then

$$\int_{\Omega_1} u(x)\Phi\left(\left|\frac{\vec{g}(x)}{L_q(x)}\right|\right)d\mu_1(x) \leq \int_{\Omega_2} W_q(y)\Phi\left(\frac{\left|\vec{f}(y)\right|}{\left\|\vec{f}(y)\right\|_q}\right)d\mu_2(y), \tag{42}$$

under the assumptions:

(i) $\frac{\vec{f}(y)}{\left\|\vec{f}(y)\right\|_q}$, $\Phi\left(\frac{\left|\vec{f}(y)\right|}{\left\|\vec{f}(y)\right\|_q}\right)$ are both $k(x,y)\left\|\vec{f}(y)\right\|_q d\mu_2(y)$ -integrable, μ_1 -a.e. in $x \in \Omega_1$,

(ii) $W_q(y)\Phi\left(\frac{\left|\vec{f}(y)\right|}{\left\|\vec{f}(y)\right\|_q}\right)$ is μ_2 -integrable.

Theorem 8 comes directly from Theorem 1.

We will also use:

Let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ measure spaces with positive σ-finite measures, and $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ are nonnegative measurable functions, with $k_i(x, \cdot)$ measurable on Ω_2, and measurable functions $g_{ji} : \Omega_1 \to \mathbb{R}$:

$$g_{ji}(x) = \int_{\Omega_2} k_i(x,y) f_{ji}(y) d\mu_2(y),$$

where $f_{ji} : \Omega_2 \to \mathbb{R}$ are measurable functions, for all $j = 1,2; i = 1,...,m$.

Theorem 9 ([3], p. 552) Here $0 < f_{2i}(y) < \infty$, a.e., $i = 1,...,m.$ Assume that the functions ($i = 1,...,m \in \mathbb{N}$)

$$x \mapsto \left(\frac{u(x)k_i(x,y)f_{2i}(y)}{g_{2i}(x)}\right)$$

are integrable on Ω_1, for each fixed $y \in \Omega_2$; with $g_{2i}(x) > 0$, a.e. on Ω_1.

Define ψ_i on Ω_2 by

$$\psi_i(y) := f_{2i}(y) \int_{\Omega_1} u(x) \frac{k_i(x,y)}{g_{2i}(x)} d\mu_1(x) < \infty, \tag{43}$$

a.e. on Ω_2.

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1.$ Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,...,m,$ be convex and increasing. Then

$$\int_{\Omega_1} u(x)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{g_{1i}(x)}{g_{2i}(x)}\right|\right)d\mu_1(x)\leq \prod_{i=1}^{m}\left(\int_{\Omega_2}\psi_i(y)\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)^{p_i}d\mu_2(y)\right)^{\frac{1}{p_i}}, \tag{44}$$

under the assumptions:

(i) $\frac{f_{1i}(y)}{f_{2i}(y)}$, $\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)^{p_i}$ are both $k_i(x,y)f_{2i}(y)d\mu_2(y)$ -integrable, μ_1 -a.e. in $x\in\Omega_1$,

(ii) $\psi_i(y)\Phi_i\left(\left|\frac{f_{1i}(y)}{f_{2i}(y)}\right|\right)^{p_i}$ is μ_2-integrable, $i=1,\ldots,m$.

3.Main Results

We make

Remark 10 Here $\rho_j,\mu_j,\gamma_j,\omega_j>0$; $f_{ji}\in C([a,b])$ and $\psi\in C^1([a,b])$ which is increasing; $j=1,\ldots,m$ and $i=1,\ldots,n$. Set

$$_{\infty}\varphi_{j+}(y):=\left\|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi}f_j(y)}\right\|_{\infty}:=\max_{\substack{j=1,\ldots,m\\ i=1,\ldots,n}}\left\{\left|e_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi}f_{ji}(y)\right|\right\}, \tag{45}$$

and

$$_{q}\varphi_{j+}(y):=\left\|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi}f_j(y)}\right\|_{q}:=\left(\sum_{i=1}^{n}\left|e_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi}f_{ji}(y)\right|^{q}\right)^{\frac{1}{q}}, q\geq 1; \tag{46}$$

$y\in[a,b]$, which $_{q}\varphi_{j+}$ are continuous functions, $j=1,\ldots,m$. We have that

$$0<{}_{q}\varphi_{j+}(y)<\infty \text{ in}[a,b], \tag{47}$$

$j=1,\ldots,m;$ where $1\leq q\leq\infty$ is fixed.

Here it is

$$k_j^{+}(x,y):=k_j(x,y)=\begin{cases}\psi'(y)(\psi(x)-\psi(y))^{\mu_j-1}E_{\rho_j,\mu_j}^{\gamma_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right], a<y\leq x,\\ 0, x<y<b,\end{cases} \tag{48}$$

$j=1,\ldots,m,$ and

$$L_{jq}^{+}(x):=\int_a^x\psi'(y)(\psi(x)-\psi(y))^{\mu_j-1}E_{\rho_j,\mu_j}^{\gamma_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right]{}_{q}\varphi_{j+}(y)dy, \tag{49}$$

$\forall$ $x\in[a,b]$, $1\leq q\leq\infty$.

We have that $L_{jq}^{+}(x) > 0$ on $[a,b]$.

Let $\rho \in \{1,...,m\}$ be fixed. The weight function u is chosen so that

$$U_m^{+}(y) := \left(\prod_{j=1}^{m} {}_q\varphi_{j+}(y)\right)\int_y^b \frac{u(x)\prod_{j=1}^{m} k_j^{+}(x,y)}{\prod_{j=1}^{m} L_{jq}^{+}(x)}dx < \infty, \tag{50}$$

$\forall\ y \in [a,b]$, and that U_m^{+} is integrable on $[a,b]$.

A direct application of Theorem 5 gives:

Theorem 11 It is all as in Remark 10. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,...,m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi} f_j(x)}\right|}{L_{jq}^{+}(x)}\right)dx \le \left(\prod_{\substack{j=1\\ j\neq\rho}}^{m}\int_a^b \Phi_j\left(\frac{\left|\overrightarrow{f_j(y)}\right|}{{}_q\varphi_{j+}(y)}\right)dy\right)\left(\int_a^b \Phi_\rho\left(\frac{\left|\overrightarrow{f_\rho(y)}\right|}{{}_q\varphi_{\rho+}(y)}\right)U_m^{+}(y)dy\right). \tag{51}$$

We make

Remark 12 Here $\rho_j,\mu_j,\gamma_j,\omega_j > 0;\ f_{ji} \in C([a,b])$ and $\psi \in C^1([a,b])$ which is increasing; $j = 1,...,m$ and $i = 1,...,n$. Set

$$_\infty\varphi_{j-}(y) := \left\|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j(y)}\right\|_\infty := \max_{\substack{j=1,...,m\\ i=1,...,n}}\left\{\left|e_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_{ji}(y)\right|\right\}, \tag{52}$$

and

$$_q\varphi_{j-}(y) := \left\|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j(y)}\right\|_q := \left(\sum_{i=1}^{n}\left|e_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_{ji}(y)\right|^q\right)^{\frac{1}{q}},\ q \ge 1; \tag{53}$$

$y \in [a,b]$, which $_q\varphi_{j-}$ are continuous functions, $j = 1,...,m$. We have also that

$$0 <_q \varphi_{j-}(y) < \infty \text{ in}[a,b], \tag{54}$$

$j = 1,...,m;$ where $1 \le q \le \infty$ is fixed.

Here it is

$$k_j^-(x,y) := k_j(x,y) = \begin{cases} \psi'(y)(\psi(y)-\psi(x))^{\mu_j-1} E_{\rho_j,\mu_j}^{\gamma_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right], x \le y < b, \\ 0, a < y < x, \end{cases} \tag{55}$$

$j=1,...,m,$ and

$$L_{jq}^-(x) := \int_x^b \psi'(y)(\psi(y)-\psi(x))^{\mu_j-1} E_{\rho_j,\mu_j}^{\gamma_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right]_q \varphi_{j-}(y)dy, \tag{56}$$

$\forall\ x \in [a,b]$, $1 \le q \le \infty$.

We have that $L_{jq}^-(x) > 0$ on $[a,b]$.

Let $\rho \in \{1,...,m\}$ be fixed. The weight function u is chosen so that

$$U_m^-(y) := \left(\prod_{j=1}^m {}_q\varphi_{j-}(y)\right)\int_a^y \frac{u(x)\prod_{j=1}^m k_j^-(x,y)}{\prod_{j=1}^m L_{jq}^-(x)} dx < \infty, \tag{57}$$

$\forall\ y \in [a,b]$, and that U_m^- is integrable on $[a,b]$.

A direct application of Theorem 5 gives:

Theorem 13 It is all as in Remark 12. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j=1,...,m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x)\prod_{j=1}^m \Phi_j\left(\frac{\left|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j(x)}\right|}{L_{jq}^-(x)}\right)dx \le \left(\prod_{\substack{j=1\\ j\neq\rho}}^m \int_a^b \Phi_j\left(\frac{\left|\overrightarrow{f_j(y)}\right|}{{}_q\varphi_{j-}(y)}\right)dy\right)\left(\int_a^b \Phi_\rho\left(\frac{\left|\overrightarrow{f_\rho(y)}\right|}{{}_q\varphi_{\rho-}(y)}\right)U_m^-(y)dy\right). \tag{58}$$

We make

Remark 14 Here $j=1,...,m;\ i=1,...,n.$ Let $\rho_j,\mu_j,\omega_j > 0,\ \gamma_j < 0,$ and $f_{ji} \in C^{N_j}([a,b]),\ N_j = \lceil \mu_j \rceil,\ \mu_j \notin \mathbb{N};$ $\theta := \max(N_1,...,N_m),\ \psi \in C^\theta([a,b]),\ \psi$ is increasing with $\psi'(x) \neq 0$ over $[a,b]$. Set $f_{ji\psi}^{[N_j]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N_j} f_{ji}(x),\ x \in [a,b]$. Set

$${}_\infty\lambda_{j+}(y) := \left\|\overrightarrow{{}^C D_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi} f_j(y)}\right\|_\infty := \max_{\substack{j=1,...,m\\ i=1,...,n}}\left\{\left|{}^C D_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi} f_{ji}(y)\right|\right\}, \tag{59}$$

and

$$ {}_q\lambda_{j+}(y) := \left\| \overrightarrow{{}^C D^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,a+} f_j(y)} \right\|_q := \left(\sum_{i=1}^{n} \left| {}^C D^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,a+} f_{ji}(y) \right|^q \right)^{\frac{1}{q}}, \; q \geq 1; \tag{60} $$

$y \in [a,b]$, which all ${}_q\lambda_{j+}$ are continuous functions, $j = 1,\ldots,m$. We also have that

$$ 0 <_q \lambda_{j+}(y) < \infty \text{ in} [a,b], \tag{61} $$

$j = 1,\ldots,m;$ where $1 \leq q \leq \infty$ is fixed.

Here it is

$$ {}^C k_j^+(x,y) := k_j(x,y) = \begin{cases} \psi'(y)(\psi(x)-\psi(y))^{N_j-\mu_j-1} E^{-\gamma_j}_{\rho_j,N_j-\mu_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right], \, a < y \leq x, \\ 0, \, x < y < b, \end{cases} \tag{62} $$

$j = 1,\ldots,m,$ and

$$ {}^C L^+_{jq}(x) := \int_a^x \psi'(y)(\psi(x)-\psi(y))^{N_j-\mu_j-1} E^{-\gamma_j}_{\rho_j,N_j-\mu_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right] {}_q\lambda_{j+}(y)dy, \tag{63} $$

$\forall \; x \in [a,b]$, $1 \leq q \leq \infty, \; j = 1,\ldots,m$.

We have that ${}^C L^+_{jq}(x) > 0$ on $[a,b]$.

Let $\rho \in \{1,\ldots,m\}$ be fixed. The weight function u is chosen so that

$$ {}^C U_m^+(y) := \left(\prod_{j=1}^{m} {}_q\lambda_{j+}(y) \right) \int_y^b \frac{u(x) \prod_{j=1}^{m} {}^C k_j^+(x,y)}{\prod_{j=1}^{m} {}^C L^+_{jq}(x)} dx < \infty, \tag{64} $$

$\forall \; y \in [a,b]$, and that ${}^C U_m^+$ is integrable on $[a,b]$.

A direct application of Theorem 11, see also (6), gives:

Theorem 15 It is all as in Remark 14. Here $\Phi_j : \mathbb{R}^n_+ \to \mathbb{R}_+$, $j = 1,\ldots,m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x)\prod_{j=1}^m \Phi_j\left(\frac{\left|\overrightarrow{{}^C D^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,a+} f_j(x)}\right|}{{}^C L^+_{jq}(x)}\right)dx \le \left(\prod_{\substack{j=1\\ j\neq\rho}}^m \int_a^b \Phi_j\left(\frac{\left\|\overrightarrow{f^{[N_j]}_{j\psi}(y)}\right\|}{{}_q\lambda_{j+}(y)}\right)dy\right)\left(\int_a^b \Phi_\rho\left(\frac{\left\|\overrightarrow{f^{[N_\rho]}_{\rho\psi}(y)}\right\|}{{}_q\lambda_{\rho+}(y)}\right)^C U^+_m(y)dy\right). \qquad (65)$$

We make

Remark 16 Here $j=1,\dots,m;\ i=1,\dots,n.$ Let $\rho_j,\mu_j,\omega_j>0,\ \gamma_j<0,$ and $f_{ji}\in C^{N_j}([a,b]),\ N_j=\lceil\mu_j\rceil,\ \mu_j\notin \mathbb{N};$ $\theta:=\max(N_1,\dots,N_m),\ \psi\in C^\theta([a,b]),\ \psi$ is increasing with $\psi'(x)\neq 0$ over $[a,b]$. Set $f^{[N_j]}_{ji\psi}(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{N_j} f_{ji}(x),\ x\in[a,b]$. Set

$$ {}_\infty\lambda_{j-}(y):=\left\|\overrightarrow{{}^C D^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_j(y)}\right\|_\infty := \max_{\substack{j=1,\dots,m\\ i=1,\dots,n}}\left\{\left|{}^C D^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_{ji}(y)\right|\right\}, \qquad (66)$$

and

$$ {}_q\lambda_{j-}(y):=\left\|\overrightarrow{{}^C D^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_j(y)}\right\|_q := \left(\sum_{i=1}^n\left|{}^C D^{\gamma_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_{ji}(y)\right|^q\right)^{\frac{1}{q}},\ q\ge 1; \qquad (67)$$

$y\in[a,b]$, which all ${}_q\lambda_{j-}$ are continuous functions, $j=1,\dots,m$. We also have that

$$0<{}_q\lambda_{j-}(y)<\infty \text{ in}[a,b], \qquad (68)$$

$j=1,\dots,m;$ where $1\le q\le\infty$ is fixed.

Here it is

$$ {}^C k_j^-(x,y):=k_j(x,y)=\begin{cases}\psi'(y)(\psi(y)-\psi(x))^{N_j-\mu_j-1}E^{-\gamma_j}_{\rho_j,N_j-\mu_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right], x\le y<b,\\ 0,\ a<y<x,\end{cases} \qquad (69)$$

$j=1,\dots,m,$ and

$$ {}^C L^-_{jq}(x):=\int_x^b \psi'(y)(\psi(y)-\psi(x))^{N_j-\mu_j-1} E^{-\gamma_j}_{\rho_j,N_j-\mu_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right]{}_q\lambda_{j-}(y)dy, \qquad (70)$$

$\forall\ x\in[a,b],\ 1\le q\le\infty,\ j=1,\dots,m$.

We have that ${}^C L^-_{jq}(x)>0$ on $[a,b]$.

Let $\rho \in \{1,...,m\}$ be fixed. The weight function u is chosen so that

$$ {}^{C}U_{m}^{-}(y) := \left(\prod_{j=1}^{m} {}_{q}\lambda_{j-}(y) \right) \int_{a}^{y} \frac{u(x) \prod_{j=1}^{m} {}^{C}k_{j}^{-}(x,y)}{\prod_{j=1}^{m} {}^{C}L_{jq}^{-}(x)} dx < \infty, \quad (71) $$

$\forall\ y \in [a,b]$, and that ${}^{C}U_{m}^{-}$ is integrable on $[a,b]$.

A direct application of Theorem 13, see also (7), gives:

Theorem 17 It is all as in Remark 16. Here $\Phi_j : \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$, $j = 1,...,m$, are convex functions increasing per coordinate. Then

$$ \int_{a}^{b} u(x) \prod_{j=1}^{m} \Phi_j \left(\frac{\left| \overrightarrow{{}^{C}D_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j(x)} \right|}{{}^{C}L_{jq}^{-}(x)} \right) dx \leq \left(\prod_{\substack{j=1 \\ j\neq\rho}}^{m} \int_{a}^{b} \Phi_j \left(\frac{\left| \overrightarrow{f_{j\psi}^{[N_j]}(y)} \right|}{{}_{q}\lambda_{j-}(y)} \right) dy \right) \left(\int_{a}^{b} \Phi_\rho \left(\frac{\left| \overrightarrow{f_{\rho\psi}^{[N_\rho]}(y)} \right|}{{}_{q}\lambda_{\rho-}(y)} \right) {}^{C}U_{m}^{-}(y) dy \right). \quad (72) $$

We make

Remark 18 Here $j = 1,...,m;\ i = 1,...,n.$ Let $\rho_j, \mu_j, \omega_j > 0,\ \gamma_j < 0,$ and $f_{ji} \in C([a,b]),\ N_j = \lceil \mu_j \rceil,\ \mu_j \notin \mathbb{N};$ $\theta := \max(N_1,...,N_m),\ \psi \in C^{\theta}([a,b]),\ \psi$ is increasing with $\psi'(x) \neq 0$ over $[a,b]$. Here $0 \leq \beta_j \leq 1$ and $\xi_j = \mu_j + \beta_j (N_j - \mu_j)$. We assume that ${}^{RL}D_{\rho_j,\xi_j,\omega_j,a+}^{\gamma_j(1-\beta_j);\psi} f_{ji} \in C([a,b])$, $j = 1,...,m,\ i = 1,...,n.$ Set

$$ {}_{\infty}M_{j+}(y) := \left\| \overrightarrow{{}^{H}\mathsf{D}_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j,\beta_j;\psi} f_j(y)} \right\|_{\infty} := \max_{\substack{j=1,...,m \\ i=1,...,n}} \left\{ \left| {}^{H}\mathsf{D}_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j,\beta_j;\psi} f_{ji}(y) \right| \right\}, \quad (73) $$

and

$$ {}_{q}M_{j+}(y) := \left\| \overrightarrow{{}^{H}\mathsf{D}_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j,\beta_j;\psi} f_j(y)} \right\|_{q} := \left(\sum_{i=1}^{n} \left| {}^{H}\mathsf{D}_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j,\beta_j;\psi} f_{ji}(y) \right|^{q} \right)^{\frac{1}{q}},\ q \geq 1; \quad (74) $$

$y \in [a,b]$, which all ${}_{q}M_{j+}$ are continuous functions, $j = 1,...,m$. We also have that

$$ 0 <_{q} M_{j+}(y) < \infty \text{ in } [a,b], \quad (75) $$

$j = 1,...,m;$ where $1 \leq q \leq \infty$ is fixed.

Here it is

$$ {}^{P}k_{j}^{+}(x,y):=k_{j}(x,y)=\begin{cases}\psi'(y)(\psi(x)-\psi(y))^{\xi_{j}-\mu_{j}-1}E_{\rho_{j},\xi_{j}-\mu_{j}}^{-\gamma_{j}\beta_{j}}\left[\omega_{j}(\psi(x)-\psi(y))^{\rho_{j}}\right], a<y\leq x,\\ 0, x<y<b,\end{cases} \tag{76}$$

$j=1,...,m,$ and

$$ {}^{P}L_{jq}^{+}(x):=\int_{a}^{x}\psi'(y)(\psi(x)-\psi(y))^{\xi_{j}-\mu_{j}-1} E_{\rho_{j},\xi_{j}-\mu_{j}}^{-\gamma_{j}\beta_{j}}\left[\omega_{j}(\psi(x)-\psi(y))^{\rho_{j}}\right]{}_{q}M_{j+}(y)dy, \tag{77}$$

$\forall\ x\in[a,b]$, $1\leq q\leq\infty$.

We have that ${}^{P}L_{jq}^{+}(x)>0$ on $[a,b]$.

Let $\overline{\rho}\in\{1,...,m\}$ be fixed. The weight function u is chosen so that

$$ {}^{P}U_{m}^{+}(y):=\left(\prod_{j=1}^{m}{}_{q}M_{j+}(y)\right)\int_{y}^{b}\frac{u(x)\prod_{j=1}^{m}{}^{P}k_{j}^{+}(x,y)}{\prod_{j=1}^{m}{}^{P}L_{jq}^{+}(x)}dx<\infty, \tag{78}$$

$\forall\ y\in[a,b]$, and that ${}^{P}U_{m}^{+}$ is integrable on $[a,b]$.

A direct application of Theorem 11, see also (15), gives:

Theorem 19 It is all as in Remark 18. Here $\Phi_{j}:\mathbb{R}_{+}^{n}\rightarrow\mathbb{R}_{+}$, $j=1,...,m$, are convex functions increasing per coordinate. Then

$$ \int_{a}^{b}u(x)\prod_{j=1}^{m}\Phi_{j}\left(\frac{\left|\overrightarrow{{}^{H}\mathsf{D}_{\rho_{j},\mu_{j},\omega_{j},a+}^{\gamma_{j},\beta_{j};\psi}f_{j}(x)}\right|}{{}^{P}L_{jq}^{+}(x)}\right)dx\leq\left(\prod_{\substack{j=1\\ j\neq\overline{\rho}}}^{m}\int_{a}^{b}\Phi_{j}\left(\frac{\left|\overrightarrow{{}^{RL}D_{\rho_{j},\xi_{j},\omega_{j},a+}^{\gamma_{j}(1-\beta_{j});\psi}f_{j}(y)}\right|}{{}_{q}M_{j+}(y)}\right)dy\right) \tag{79}$$

$$ \left(\int_{a}^{b}\Phi_{\overline{\rho}}\left(\frac{\left|\overrightarrow{{}^{RL}D_{\rho_{j},\xi_{j},\omega_{j},a+}^{\gamma_{j}(1-\beta_{j});\psi}f_{\overline{\rho}}(y)}\right|}{{}_{q}M_{\overline{\rho}+}(y)}\right){}^{P}U_{m}^{+}(y)dy\right).$$

We make

Remark 20 Here $j=1,...,m;\ i=1,...,n.$ Let $\rho_j, \mu_j, \omega_j > 0,\ \gamma_j < 0,$ and $f_{ji} \in C([a,b]),\ N_j = \lceil \mu_j \rceil,\ \mu_j \notin \mathbb{N};$ $\theta := \max(N_1,...,N_m),\ \psi \in C^{\theta}([a,b]),\ \psi$ is increasing with $\psi'(x) \neq 0$ over $[a,b]$. Here $0 \leq \beta_j \leq 1$ and $\xi_j = \mu_j + \beta_j(N_j - \mu_j)$. We assume that ${}^{RL}D^{\gamma_j(1-\beta_j);\psi}_{\rho_j,\xi_j,\omega_j,b-} f_{ji} \in C([a,b])$, $j=1,...,m,\ i=1,...,n.$ Set

$$ {}_{\infty}M_{j-}(y) := \left\| \overrightarrow{{}^{H}\mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_j(y)} \right\|_{\infty} := \max_{\substack{j=1,...,m \\ i=1,...,n}} \left\{ \left| {}^{H}\mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_{ji}(y) \right| \right\}, \tag{80} $$

and

$$ {}_{q}M_{j-}(y) := \left\| \overrightarrow{{}^{H}\mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_j(y)} \right\|_{q} := \left(\sum_{i=1}^{n} \left| {}^{H}\mathsf{D}^{\gamma_j,\beta_j;\psi}_{\rho_j,\mu_j,\omega_j,b-} f_{ji}(y) \right|^{q} \right)^{\frac{1}{q}},\ q \geq 1; \tag{81} $$

$y \in [a,b]$, which all ${}_{q}M_{j-}$ are continuous functions, $j=1,...,m$. We also have that

$$ 0 <_{q} M_{j-}(y) < \infty \text{ in} [a,b], \tag{82} $$

$j=1,...,m;$ where $1 \leq q \leq \infty$ is fixed.

Here it is

$$ {}^{P}k_j^{-}(x,y) := k_j(x,y) = \begin{cases} \psi'(y)(\psi(y)-\psi(x))^{\xi_j-\mu_j-1} E^{-\gamma_j\beta_j}_{\rho_j,\xi_j-\mu_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right], x \leq y < b, \\ 0,\ a < y < x, \end{cases} \tag{83} $$

$j=1,...,m,$ and

$$ {}^{P}L_{jq}^{-}(x) := \int_x^b \psi'(y)(\psi(y)-\psi(x))^{\xi_j-\mu_j-1} \quad E^{-\gamma_j\beta_j}_{\rho_j,\xi_j-\mu_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right]{}_{q}M_{j-}(y)dy, \tag{84} $$

$\forall\ x \in [a,b]$, $1 \leq q \leq \infty$.

We have that ${}^{P}L_{jq}^{-}(x) > 0$ on $[a,b]$.

Let $\overline{\rho} \in \{1,...,m\}$ be fixed. The weight function u is chosen so that

$$ {}^{P}U_m^{-}(y) := \left(\prod_{j=1}^{m} {}_{q}M_{j-}(y) \right) \int_a^y \frac{u(x) \prod_{j=1}^{m} {}^{P}k_j^{-}(x,y)}{\prod_{j=1}^{m} {}^{P}L_{jq}^{-}(x)} dx < \infty, \tag{85} $$

$\forall\ y \in [a,b]$, and that ${}^{P}U_m^{-}$ is integrable on $[a,b]$.

A direct application of Theorem 13, see also (16), gives:

Theorem 21 It is all as in Remark 20. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,\ldots,m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x)\prod_{j=1}^m \Phi_j\left(\frac{\left|\overrightarrow{{}^{H}\mathrm{D}_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j,\beta_j;\psi} f_j(x)}\right|}{{}^{P}L_{jq}^{-}(x)}\right)dx \leq \left(\prod_{\substack{j=1\\ j\neq\rho}}^{m}\int_a^b \Phi_j\left(\frac{\left|\overrightarrow{{}^{RL}D_{\rho_j,\xi_j,\omega_j,b-}^{\gamma_j(1-\beta_j);\psi} f_j(y)}\right|}{{}_{q}M_{j-}(y)}\right)dy\right) \tag{86}$$

$$\left(\int_a^b \Phi_{\bar{\rho}}\left(\frac{\left|\overrightarrow{{}^{RL}D_{\rho_j,\xi_j,\omega_j,b-}^{\gamma_j(1-\beta_j);\psi} f_{\bar{\rho}}(y)}\right|}{{}_{q}M_{\bar{\rho}-}(y)}\right){}^{P}U_m^{-}(y)dy\right).$$

We make

Remark 22 The basic background here is as in Remark 10. Also ${}_{q}\varphi_{j+}(y)$, $1 \leq q \leq \infty$, $y \in [a,b]$ is as in (45), (46), (47); $k_j^{+}(x,y)$ is as (48) and $L_{jq}^{+}(x)$ as in (49), where $x, y \in [a,b]$. Here it is

$$K_j^{+}(x) := K_j(x) = (\psi(x)-\psi(a))^{\mu_j} E_{\rho_j,\mu_j+1}^{\gamma_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right], \tag{87}$$

$\forall\ x \in [a,b]$, $j = 1,\ldots,m$. Indeed it is

$$\frac{k_j^{+}(x,y)}{K_j^{+}(x)} = \left(\chi_{(a,x]}(y)\psi'(y)\frac{(\psi(x)-\psi(y))^{\mu_j-1}}{(\psi(x)-\psi(a))^{\mu_j}}\right)\left(\frac{E_{\rho_j,\mu_j}^{\gamma_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right]}{E_{\rho_j,\mu_j+1}^{\gamma_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right]}\right), \tag{88}$$

$\forall\ x, y \in [a,b]$, $j = 1,\ldots,m;$ χ is the characteristic function.

We define ${}_{q}W_{j+}$ on $[a,b]$, with appropiate choice of weight function u, by

$${}_{q}W_{j+}(y) := {}_{q}\varphi_{j+}(y)\left(\int_y^b \frac{u(x)k_j^{+}(x,y)}{K_j^{+}(x)}dx\right) < \infty, \tag{89}$$

$\forall\ y \in [a,b]$, and that ${}_{q}W_{j+}$ is integrable on $[a,b]$; $j = 1,\ldots,m.$

A direct application of Theorem 6, see also (2), follows:

Theorem 23 It is all as in Remark 22. Let $p_j > 1 : \sum_{j=1}^{m}\frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,\ldots,m$, be convex and increasing per coordinate. Then

$$\int_a^b u(x)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j;\psi}f_j(x)}\right|}{L_{jq}^{+}(x)}\right)dx \leq \prod_{j=1}^{m}\left(\int_a^b {}_qW_{j+}(y)\Phi_j\left(\frac{\left|\overrightarrow{f_j(y)}\right|}{{}_q\varphi_{j+}(y)}\right)^{p_j}dy\right)^{\frac{1}{p_j}}. \tag{90}$$

We make

Remark 24 The basic background here is as in Remark 12. Also ${}_q\varphi_{j-}(y)$, $1 \leq q \leq \infty$, $y \in [a,b]$ is as in (52), (53), (54); $k_j^{-}(x,y)$ is as (55) and $L_{jq}^{-}(x)$ as in (56), where $x, y \in [a,b]$. Here it is

$$K_j^{-}(x) := K_j(x) = (\psi(b)-\psi(x))^{\mu_j} E_{\rho_j,\mu_j+1}^{\gamma_j}\left[\omega_j(\psi(b)-\psi(x))^{\rho_j}\right], \tag{91}$$

$\forall\ x \in [a,b]$, $j = 1,...,m$. Indeed it is

$$\frac{k_j^{-}(x,y)}{K_j^{-}(x)} = \left(\chi_{[x,b)}(y)\psi'(y)\frac{(\psi(y)-\psi(x))^{\mu_j-1}}{(\psi(b)-\psi(x))^{\mu_j}}\right)\left(\frac{E_{\rho_j,\mu_j}^{\gamma_j}\left[\omega_j(\psi(y)-\psi(x))^{\rho_j}\right]}{E_{\rho_j,\mu_j+1}^{\gamma_j}\left[\omega_j(\psi(b)-\psi(x))^{\rho_j}\right]}\right), \tag{92}$$

$\forall\ x, y \in [a,b]$, $j = 1,...,m$.

We define ${}_qW_{j-}$ on $[a,b]$, with appropiate choice of weight function u, by

$${}_qW_{j-}(y) := {}_q\varphi_{j-}(y)\left(\int_a^y \frac{u(x)k_j^{-}(x,y)}{K_j^{-}(x)}dx\right) < \infty, \tag{93}$$

$\forall\ y \in [a,b]$, and that ${}_qW_{j-}$ is integrable on $[a,b]$; $j = 1,...,m$.

A direct application of Theorem 6, see also (3), follows:

Theorem 25 It is all as in Remark 24. Let $p_j > 1 : \sum_{j=1}^{m}\frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,...,m$, be convex and increasing per coordinate. Then

$$\int_a^b u(x)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|\overrightarrow{e_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi}f_j(x)}\right|}{L_{jq}^{-}(x)}\right)dx \leq \prod_{j=1}^{m}\left(\int_a^b {}_qW_{j-}(y)\Phi_j\left(\frac{\left|\overrightarrow{f_j(y)}\right|}{{}_q\varphi_{j-}(y)}\right)^{p_j}dy\right)^{\frac{1}{p_j}}. \tag{94}$$

We need

Remark 26 The basic background here is as in Remark 14. Also ${}_q\lambda_{j+}(y)$, $1 \leq q \leq \infty$, $y \in [a,b]$ is as in (59), (60), (61); ${}^{C}k_j^{+}(x,y)$ is as (62) and ${}^{C}L_{jq}^{+}(x)$ as in (63), where $x, y \in [a,b]$. Here it is

$$ {}^{C}K_{j}^{+}(x) := K_{j}(x) = (\psi(x)-\psi(a))^{N_{j}-\mu_{j}} E_{\rho_{j},N_{j}-\mu_{j}+1}^{-\gamma_{j}}\left[\omega_{j}(\psi(x)-\psi(a))^{\rho_{j}}\right], \tag{95} $$

$\forall\ x \in [a,b]$, $j = 1,\ldots,m$. Indeed it is

$$ \frac{{}^{C}k_{j}^{+}(x,y)}{{}^{C}K_{j}^{+}(x)} = \left(\chi_{(a,x]}(y)\psi'(y)\frac{(\psi(x)-\psi(y))^{N_{j}-\mu_{j}-1}}{(\psi(x)-\psi(a))^{N_{j}-\mu_{j}}}\right)\left(\frac{E_{\rho_{j},N_{j}-\mu_{j}}^{-\gamma_{j}}\left[\omega_{j}(\psi(x)-\psi(y))^{\rho_{j}}\right]}{E_{\rho_{j},N_{j}-\mu_{j}+1}^{-\gamma_{j}}\left[\omega_{j}(\psi(x)-\psi(a))^{\rho_{j}}\right]}\right), \tag{96} $$

$\forall\ x,y \in [a,b]$, $j = 1,\ldots,m$.

We define ${}_{q}^{C}W_{j+}$ on $[a,b]$, with appropiate choice of weight function u, by

$$ {}_{q}^{C}W_{j+}(y) := {}_{q}\lambda_{j+}(y)\left(\int_{y}^{b}\frac{u(x)\,{}^{C}k_{j}^{+}(x,y)}{{}^{C}K_{j}^{+}(x)}dx\right) < \infty, \tag{97} $$

$\forall\ y \in [a,b]$, and that ${}_{q}^{C}W_{j+}$ is integrable on $[a,b]$; $j = 1,\ldots,m.$

A direct application of Theorem 23, see also (6), follows:

Theorem 27 It is all as in Remark 26. Let $p_{j} > 1 : \sum_{j=1}^{m}\frac{1}{p_{j}} = 1$. Let the functions $\Phi_{j} : \mathbb{R}_{+}^{n} \to \mathbb{R}_{+}$, $j = 1,\ldots,m$, be convex and increasing per coordinate. Then

$$ \int_{a}^{b}u(x)\prod_{j=1}^{m}\Phi_{j}\left(\frac{\left|\overrightarrow{{}^{C}D_{\rho_{j},\mu_{j},\omega_{j},a+}^{\gamma_{j};\psi}f_{j}(x)}\right|}{{}^{C}L_{jq}^{+}(x)}\right)dx \leq \prod_{j=1}^{m}\left(\int_{a}^{b}{}_{q}^{C}W_{j+}(y)\Phi_{j}\left(\frac{\left|\overrightarrow{f_{j\psi}^{[N_{j}]}(y)}\right|}{{}_{q}\lambda_{j+}(y)}\right)^{p_{j}}dy\right)^{\frac{1}{p_{j}}}. \tag{98} $$

We need

Remark 28 The basic background here is as in Remark 16. Also ${}_{q}\lambda_{j-}(y)$, $1 \leq q \leq \infty$, $y \in [a,b]$ is as in (66), (67), (68); ${}^{C}k_{j}^{-}(x,y)$ is as (69) and ${}^{C}L_{jq}^{-}(x)$ as in (70), where $x,y \in [a,b]$. Here it is

$$ {}^{C}K_{j}^{-}(x) := K_{j}(x) = (\psi(b)-\psi(x))^{N_{j}-\mu_{j}} E_{\rho_{j},N_{j}-\mu_{j}+1}^{-\gamma_{j}}\left[\omega_{j}(\psi(b)-\psi(x))^{\rho_{j}}\right], \tag{99} $$

$\forall\ x \in [a,b]$, $j = 1,\ldots,m$. Indeed it is

$$ \frac{{}^{C}k_{j}^{-}(x,y)}{{}^{C}K_{j}^{-}(x)} = \left(\chi_{[x,b)}(y)\psi'(y)\frac{(\psi(y)-\psi(x))^{N_{j}-\mu_{j}-1}}{(\psi(b)-\psi(x))^{N_{j}-\mu_{j}}}\right)\left(\frac{E_{\rho_{j},N_{j}-\mu_{j}}^{-\gamma_{j}}\left[\omega_{j}(\psi(y)-\psi(x))^{\rho_{j}}\right]}{E_{\rho_{j},N_{j}-\mu_{j}+1}^{-\gamma_{j}}\left[\omega_{j}(\psi(b)-\psi(x))^{\rho_{j}}\right]}\right), \tag{100} $$

$\forall\ x,y \in [a,b]$, $j = 1,\ldots,m.$

We define ${}_q^C W_{j-}$ on $[a,b]$, with appropiate choice of weight function u, by

$$ {}_q^C W_{j-}(y) := {}_q\lambda_{j-}(y)\left(\int_a^y \frac{u(x)^C k_j^-(x,y)}{{}^C K_j^-(x)}dx\right) < \infty, \tag{101}$$

$\forall\ y \in [a,b]$, and that ${}_q^C W_{j-}$ is integrable on $[a,b]$; $j = 1,\ldots,m.$

A direct application of Theorem 25, see also (7), follows:

Theorem 29 It is all as in Remark 28. Let $p_j > 1 : \sum_{j=1}^{m} \frac{1}{p_j} = 1$. Let the functions $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,\ldots,m$, be convex and increasing per coordinate. Then

$$\int_a^b u(x) \prod_{j=1}^{m} \Phi_j\left(\frac{\left|\overrightarrow{{}^C D_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j;\psi} f_j(x)}\right|}{{}^C L_{jq}^-(x)}\right)dx \le \prod_{j=1}^{m}\left(\int_a^b {}_q^C W_{j-}(y)\Phi_j\left(\frac{\left|\overrightarrow{f_{j\psi}^{[N_j]}(y)}\right|}{{}_q\lambda_{j-}(y)}\right)^{p_j} dy\right)^{\frac{1}{p_j}}. \tag{102}$$

We need

Remark 30 The basic background here is as in Remark 18. Also ${}_q M_{j+}(y)$, $1 \le q \le \infty$, $y \in [a,b]$ is as in (73), (74), (75); ${}^P k_j^+(x,y)$ is as (76) and ${}^P L_{jq}^+(x)$ as in (77), where $x, y \in [a,b]$. Here it is

$$ {}^P K_j^+(x) := K_j(x) = (\psi(x) - \psi(a))^{\xi_j - \mu_j} E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j\beta_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right], \tag{103}$$

$\forall\ x \in [a,b]$, $j = 1,\ldots,m$. Indeed it is

$$\frac{{}^P k_j^+(x,y)}{{}^P K_j^+(x)} = \left(\chi_{(a,x]}(y)\psi'(y)\frac{(\psi(x)-\psi(y))^{\xi_j-\mu_j-1}}{(\psi(x)-\psi(a))^{\xi_j-\mu_j}}\right)\left(\frac{E_{\rho_j,\xi_j-\mu_j}^{-\gamma_j\beta_j}\left[\omega_j(\psi(x)-\psi(y))^{\rho_j}\right]}{E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j\beta_j}\left[\omega_j(\psi(x)-\psi(a))^{\rho_j}\right]}\right), \tag{104}$$

$\forall\ x, y \in [a,b]$, $j = 1,\ldots,m$.

We define ${}_q^P W_{j+}$ on $[a,b]$, with appropiate choice of weight function u,

$$ {}_q^P W_{j+}(y) := {}_q M_{j+}(y)\left(\int_y^b \frac{u(x)^P k_j^+(x,y)}{{}^P K_j^+(x)}dx\right) < \infty, \tag{105}$$

$\forall\ y \in [a,b]$, and that ${}_q^P W_{j+}$ is integrable on $[a,b]$; $j = 1,\ldots,m.$

A direct application of Theorem 23, see also (15), follows:

Theorem 31 It is all as in Remark 30. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,...,m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left(\frac{\left| \overleftarrow{{}^H \mathsf{D}_{\rho_j,\mu_j,\omega_j,a+}^{\gamma_j,\beta_j;\psi} f_j(x)} \right|}{{}^P L_{jq}^+(x)} \right) dx \leq \prod_{j=1}^m \left(\int_a^b {}_q^P W_{j+}(y) \Phi_j \left(\frac{\left| \overleftarrow{{}^{RL} D_{\rho_j,\xi_j,\omega_j,a+}^{\gamma_j(1-\beta_j);\psi} f_j(y)} \right|}{{}_q M_{j+}(y)} \right)^{p_j} dy \right)^{\frac{1}{p_j}}. \tag{106}$$

We need

Remark 32 The basic background here is as in Remark 20. Also ${}_q M_{j-}(y)$, $1 \leq q \leq \infty$, $y \in [a,b]$ is as in (80), (81), (82); ${}^P k_j^-(x,y)$ is as in (83) and ${}^P L_{jq}^-(x)$ as in (84), where $x, y \in [a,b]$. Here it is

$${}^P K_j^-(x) := K_j(x) = (\psi(b) - \psi(x))^{\xi_j - \mu_j} E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j \beta_j} \left[\omega_j (\psi(b) - \psi(x))^{\rho_j} \right], \tag{107}$$

$\forall$ $x \in [a,b]$, $j = 1,...,m$. Indeed it is

$$\frac{{}^P k_j^-(x,y)}{{}^P K_j^-(x)} = \left(\chi_{x,b)}(y) \psi'(y) \frac{(\psi(y) - \psi(x))^{\xi_j - \mu_j - 1}}{(\psi(b) - \psi(x))^{\xi_j - \mu_j}} \right) \left(\frac{E_{\rho_j,\xi_j-\mu_j}^{-\gamma_j \beta_j} \left[\omega_j (\psi(y) - \psi(x))^{\rho_j} \right]}{E_{\rho_j,\xi_j-\mu_j+1}^{-\gamma_j \beta_j} \left[\omega_j (\psi(b) - \psi(x))^{\rho_j} \right]} \right), \tag{108}$$

$\forall$ $x, y \in [a,b]$, $j = 1,...,m$.

We define ${}_q^P W_{j-}$ on $[a,b]$, with appropiate choice of weight function u,

$${}_q^P W_{j-}(y) := {}_q M_{j-}(y) \left(\int_a^y \frac{u(x) {}^P k_j^-(x,y)}{{}^P K_j^-(x)} dx \right) < \infty, \tag{109}$$

$\forall$ $y \in [a,b]$, and that ${}_q^P W_{j-}$ is integrable on $[a,b]$; $j = 1,...,m.$

A direct application of Theorem 25, see also (16), follows:

Theorem 33 It is all as in Remark 32. Here $\Phi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, $j = 1,...,m$, are convex functions increasing per coordinate. Then

$$\int_a^b u(x) \prod_{j=1}^m \Phi_j \left(\frac{\left| \overrightarrow{{}^H \mathsf{D}_{\rho_j,\mu_j,\omega_j,b-}^{\gamma_j,\beta_j;\psi} f_j(x)} \right|}{{}^P L_{jq}^-(x)} \right) dx \leq \prod_{j=1}^m \left(\int_a^b {}_q^P W_{j-}(y) \Phi_j \left(\frac{\left| \overrightarrow{{}^{RL} D_{\rho_j,\xi_j,\omega_j,b-}^{\gamma_j(1-\beta_j);\psi} f_j(y)} \right|}{{}_q M_{j-}(y)} \right)^{p_j} dy \right)^{\frac{1}{p_j}}. \tag{110}$$

We make

Remark 34 Let $f_i \in C([a,b])$, $i = 1,\dots,n$, and $\overrightarrow{f} = (f_1,\dots,f_n)$. We set

$$\left\| \overrightarrow{f}(y) \right\|_{\infty} := \max\{|f_1(y)|,\dots,|f_n(y)|\},$$
and
$$\left\| \overrightarrow{f}(y) \right\|_{q} := \left(\sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, \; q \geq 1; \; y \in [a,b]. \tag{111}$$

Clearly it is $\left\| \overrightarrow{f}(y) \right\|_q \in C([a,b])$, for all $1 \leq q \leq \infty$. We assume that $\left\| \overrightarrow{f}(y) \right\|_q > 0$, a.e. on (a,b), for $q \in [1,\infty]$ being fixed.

Let

$$L_q^+(x) := \int_a^x k^+(x,y) \left\| \overrightarrow{f}(y) \right\|_q dy, \; x \in [a,b], \tag{112}$$

$1 \leq q \leq \infty$ fixed.

We assume $L_q^+(x) > 0$ a.e. on (a,b).

Here we considered

$$k^+(x,y) := k(x,y) := \begin{cases} \psi'(y)(\psi(x) - \psi(y))^{\mu-1} E_{\rho,\mu}^{\gamma}\left[\omega(\psi(x) - \psi(y))^{\rho}\right], \; a < y \leq x, \\ \\ 0, \; x < y < b, \end{cases} \tag{113}$$

where $\rho, \mu, \gamma, \omega > 0$; $\psi \in C^1([a,b])$ which is increasing.

The weight function u is chosen so that

$$W_q^+(y) := \left\| \overrightarrow{f}(y) \right\|_q \left(\int_y^b \frac{u(x) k^+(x,y)}{L_q^+(x)} dx \right) < \infty, \tag{114}$$

a.e. on (a,b) and that W_q^+ is integrable on $[a,b]$.

A direct application of Theorem 8 produces:

Theorem 35 Let all as in Remark 34. Here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is a convex and increasing per coordinate function. Then

$$\int_a^b u(x)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,\omega,a+}^{\gamma;\psi} f(x)}\right|}{L_q^+(x)}\right)dx \le \int_a^b W_q^+(y)\Phi\left(\frac{\left|\overrightarrow{f}(y)\right|}{\left\|\overrightarrow{f}(y)\right\|_q}\right)dy. \quad (115)$$

We make

Remark 36 Let $f_i \in C([a,b])$, $i=1,\dots,n$, and $\overrightarrow{f}=(f_1,\dots,f_n)$. We set

$$\left\|\overrightarrow{f}(y)\right\|_\infty := \max\{|f_1(y)|,\dots,|f_n(y)|\},$$
$$\text{and}$$
$$\left\|\overrightarrow{f}(y)\right\|_q := \left(\sum_{i=1}^n |f_i(y)|^q\right)^{\frac{1}{q}}, q \ge 1; y \in [a,b]. \quad (116)$$

Clearly it is $\left\|\overrightarrow{f}(y)\right\|_q \in C([a,b])$, for all $1 \le q \le \infty$. We assume that $\left\|\overrightarrow{f}(y)\right\|_q > 0$, a.e. on (a,b), for $q \in [1,\infty]$ being fixed.

Let

$$L_q^-(x) := \int_x^b k^-(x,y)\left\|\overrightarrow{f}(y)\right\|_q dy, \; x \in [a,b], \quad (117)$$

$1 \le q \le \infty$ fixed.

We assume $L_q^-(x) > 0$ a.e. on (a,b).

Here we considered

$$k^-(x,y) := k(x,y) := \begin{cases} \psi'(y)(\psi(y)-\psi(x))^{\mu-1} E_{\rho,\mu}^{\gamma}\left[\omega(\psi(y)-\psi(x))^\rho\right], x \le y < b, \\ 0, \; a < y < x, \end{cases} \quad (118)$$

where $\rho,\mu,\gamma,\omega > 0$; $\psi \in C^1([a,b])$ which is increasing.

The weight function u is chosen so that

$$W_q^-(y) := \left\|\overrightarrow{f}(y)\right\|_q \left(\int_a^y \frac{u(x)k^-(x,y)}{L_q^-(x)}dx\right) < \infty, \quad (119)$$

a.e. on (a,b) and that W_q^- is integrable on $[a,b]$.

A direct application of Theorem 8 produces:

Theorem 37 Let all as in Remark 36. Here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is a convex and increasing per coordinate function. Then

$$\int_a^b u(x)\Phi\left(\frac{\left|\overrightarrow{e^{\gamma;\psi}_{\rho,\mu,\omega,b-}f(x)}\right|}{L_q^-(x)}\right)dx \le \int_a^b W_q^-(y)\Phi\left(\frac{\left|\overrightarrow{f}(y)\right|}{\left\|\overrightarrow{f}(y)\right\|_q}\right)dy. \tag{120}$$

Next we deal with the spherical shell:

Background 38 *We need:*

Let $N \ge 2$, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ the unit sphere on $\mathbb{R}^N$, where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. Also denote the ball $B(0,R) := \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $R > 0$, and the spherical shell

$$A := B(0,R_2) - \overline{B(0,R_1)},\ 0 < R_1 < R_2. \tag{121}$$

For the following see [12, pp. 149-150], and [13, pp. 87-88].

For $x \in \mathbb{R}^N - \{0\}$ we can write uniquely $x = r\omega$, where $r = |x| > 0$, and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.

Clearly here

$$\mathbb{R}^N - \{0\} = (0,\infty) \times S^{N-1}, \tag{122}$$

and

$$\overline{A} = [R_1, R_2] \times S^{N-1}. \tag{123}$$

We will be using

Theorem 39 ([1, p. 322]) Let $f : A \to \mathbb{R}$ be a Lebesgue integrable function. Then

$$\int_A f(x)dx = \int_{S^{N-1}}\left(\int_{R_1}^{R_2} f(r\omega)r^{N-1}dr\right)d\omega. \tag{124}$$

So we are able to write an integral on the shell in polar form using the polar coordinates (r,ω).

We need

Definition 40 Let $\rho,\mu,\gamma,w > 0$; $f \in C(\overline{A})$ and $\psi \in C^1([R_1,R_2])$ which is increasing. The left and right radial Prabhakar fractional integrals with respect to ψ are defined as follows:

$$\left(e^{\gamma;\psi}_{\rho,\mu,w,R_1+}f\right)(x) = \int_{R_1}^{r} \psi'(t)(\psi(r)-\psi(t))^{\mu-1}E^{\gamma}_{\rho,\mu}\left[w(\psi(r)-\psi(t))^{\rho}\right]f(t\omega)dt, \tag{125}$$

and

$$\left(e^{\gamma;\psi}_{\rho,\mu,w,R_2-}f\right)(x) = \int_{r}^{R_2} \psi'(t)(\psi(t)-\psi(r))^{\mu-1}E^{\gamma}_{\rho,\mu}\left[w(\psi(t)-\psi(r))^{\rho}\right]f(t\omega)dt, \tag{126}$$

where $x \in \overline{A}$, that is $x = r\omega$, $r \in [R_1,R_2]$, $\omega \in S^{N-1}$.

Based on [1], p. 288 and [2, 4], we have that (125), (126) are continuous functions over $\overline{A}$ when $\mu \geq 1$.

We make

Remark 41 Let $f_i \in C(\overline{A})$, where the shell A is as in (121), $i = 1, \ldots, n$, and $\overrightarrow{f} = (f_1, \ldots, f_n)$. We set

$$\left\| \overrightarrow{f}(y) \right\|_\infty := \max\{|f_1(y)|, \ldots, |f_n(y)|\},$$
and
$$\left\| \overrightarrow{f}(y) \right\|_q := \left(\sum_{i=1}^{n} |f_i(y)|^q \right)^{\frac{1}{q}}, q \geq 1; y \in \overline{A}. \tag{127}$$

Clearly it is $\left\| \overrightarrow{f}(y) \right\|_q \in C(\overline{A})$, $1 \leq q \leq \infty$. One can write that

$$\left\| \overrightarrow{f}(y) \right\|_q = \left\| \overrightarrow{f}(t\omega) \right\|_q, 1 \leq q \leq \infty, \tag{128}$$

where $t \in [R_1, R_2]$, $\omega \in S^{N-1}$; $y = t\omega$, by Background 38.

We assume that $\left\| \overrightarrow{f}(y) \right\|_q > 0$ on $\overline{A}$, $1 \leq q \leq \infty$ fixed.

Consider the kernel

$$k_*^+(r,t) := k(r,t) := \chi_{(R_1,r]}(t)\psi'(t)(\psi(r) - \psi(t))^{\mu-1} E_{\rho,\mu}^{\gamma}\left[w(\psi(r) - \psi(t))^{\rho}\right], \tag{129}$$

where $\rho, \mu, \gamma, w > 0$; $\psi \in C^1([R_1, R_2])$ which is increasing.

Let

$$L_{q*}^+(x) = L_{q*}^+(r\omega) = \int_{R_1}^{R_2} k_*^+(r,t) \left\| \overrightarrow{f(t\omega)} \right\|_q dt, \tag{130}$$

$x = r\omega \in \overline{A}$, $1 \leq q \leq \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We have that $L_{q*}^+(r\omega) > 0$ for $r \in (R_1, R_2]$, for every $\omega \in S^{N-1}$.

Here we choose the weight $u(x) = u(r\omega) = L_{q*}^+(r\omega)$.

Consider the function

$$W_{q*}^+(y) = W_{q*}^+(t\omega) = \left\| \overrightarrow{f(t\omega)} \right\|_q \left(\int_{R_1}^{R_2} k_*^+(r,t) dr \right) < \infty, \tag{131}$$

$\forall\ t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and $W_{q*}^+(t\omega)$ is integrable over $[R_1, R_2]$, $\forall\ \omega \in S^{N-1}$.

Here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is a convex and increasing per coordinate function. By (115) we obtain

$$\int_{R_1}^{R_2} L_{q*}^{+}(r\omega)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_1+}^{\gamma;\psi} f(r\omega)}\right|}{L_{q*}^{+}(r\omega)}\right) dr \leq \int_{R_1}^{R_2} W_{q*}^{+}(t\omega)\Phi\left(\frac{\left|\overrightarrow{f(t\omega)}\right|}{\left\|\overrightarrow{f(t\omega)}\right\|_q}\right) dt, \tag{132}$$

$\forall\ \omega \in S^{N-1}$.

Here we have $R_1 \leq r \leq R_2$, and $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$, and $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$, also $r^{N-1} r^{1-N} = 1$. Thus by (132), we have

$$\int_{R_1}^{R_2} L_{q*}^{+}(r\omega)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_1+}^{\gamma;\psi} f(r\omega)}\right|}{L_{q*}^{+}(r\omega)}\right) r^{N-1} dr \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_{R_1}^{R_2} W_{q*}^{+}(r\omega)\Phi\left(\frac{\left|\overrightarrow{f(r\omega)}\right|}{\left\|\overrightarrow{f(r\omega)}\right\|_q}\right) r^{N-1} dr, \tag{133}$$

$\forall\ \omega \in S^{N-1}$.

Therefore it holds

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2} L_{q*}^{+}(r\omega)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_1+}^{\gamma;\psi} f(r\omega)}\right|}{L_{q*}^{+}(r\omega)}\right) r^{N-1} dr\right) d\omega \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_{S^{N-1}}\left(\int_{R_1}^{R_2} W_{q*}^{+}(r\omega)\Phi\left(\frac{\left|\overrightarrow{f(r\omega)}\right|}{\left\|\overrightarrow{f(r\omega)}\right\|_q}\right) r^{N-1} dr\right) d\omega. \tag{134}$$

Using Theorem 39 we derive:

Theorem 42 All as in Remark 41. Then

$$\int_A L_{q*}^{+}(x)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_1+}^{\gamma;\psi} f(x)}\right|}{L_{q*}^{+}(x)}\right) dx \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_A W_{q*}^{+}(x)\Phi\left(\frac{\left|\overrightarrow{f(x)}\right|}{\left\|\overrightarrow{f(x)}\right\|_q}\right) dx, \tag{135}$$

where $\overrightarrow{e_{\rho,\mu,w,R_1+}^{\gamma;\psi} f(x)} = \left(\left(e_{\rho,\mu,w,R_1+}^{\gamma;\psi} f_1\right)(x),\ldots,\left(e_{\rho,\mu,w,R_1+}^{\gamma;\psi} f_n\right)(x)\right)$ and coordinates are assumed to be continuous functions on $\overline{A}$.

We make

Remark 43 Let $f_i \in C(\overline{A})$, where the shell A is as in (121), $i = 1,\ldots,n$, and $\vec{f} = (f_1,\ldots,f_n)$. We set

$$\left\|\vec{f}(y)\right\|_\infty := \max\{|f_1(y)|,\ldots,|f_n(y)|\},$$
and
$$\left\|\vec{f}(y)\right\|_q := \left(\sum_{i=1}^{n}|f_i(y)|^q\right)^{\frac{1}{q}},\ q \geq 1;\ y \in \overline{A}. \tag{136}$$

Clearly it is $\left\|\vec{f}(y)\right\|_q \in C(\overline{A})$, $1 \leq q \leq \infty$. One can write that

$$\left\|\overrightarrow{f}(y)\right\|_q = \left\|\overrightarrow{f}(t\omega)\right\|_q, 1 \leq q \leq \infty, \tag{137}$$

where $t \in [R_1, R_2]$, $\omega \in S^{N-1}$; $y = t\omega$, by Background 38.

We assume that $\left\|\overrightarrow{f}(y)\right\|_q > 0$ on $\overline{A}$, $1 \leq q \leq \infty$ fixed.

Consider the kernel

$$k_*^-(r,t) := k(r,t) := \chi_{[r,R_2)}(t)\psi'(t)(\psi(t)-\psi(r))^{\mu-1} E_{\rho,\mu}^{\gamma}\left[w(\psi(t)-\psi(r))^{\rho}\right], \tag{138}$$

where $\rho, \mu, \gamma, w > 0$; $\psi \in C^1([R_1, R_2])$ which is increasing.

Let

$$L_{q*}^-(x) = L_{q*}^-(r\omega) = \int_{R_1}^{R_2} k_*^-(r,t)\left\|\overrightarrow{f(t\omega)}\right\|_q dt, \tag{139}$$

$x = r\omega \in \overline{A}$, $1 \leq q \leq \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We have that $L_{q*}^-(r\omega) > 0$ for $r \in (R_1, R_2]$, for every $\omega \in S^{N-1}$.

Here we choose the weight $u(x) = u(r\omega) = L_{q*}^-(r\omega)$.

Consider the function

$$W_{q*}^-(y) = W_{q*}^-(t\omega) = \left\|\overrightarrow{f(t\omega)}\right\|_q \left(\int_{R_1}^{R_2} k_*^-(r,t)dr\right) < \infty, \tag{140}$$

$\forall\ t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and $W_{q*}^-(t\omega)$ is integrable over $[R_1, R_2]$, $\forall\ \omega \in S^{N-1}$.

Here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is a convex and increasing per coordinate function. By (120) we obtain

$$\int_{R_1}^{R_2} L_{q*}^-(r\omega)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_2-}^{\gamma;\psi} f(r\omega)}\right|}{L_{q*}^-(r\omega)}\right)dr \leq \int_{R_1}^{R_2} W_{q*}^-(t\omega)\Phi\left(\frac{\left|\overrightarrow{f(t\omega)}\right|}{\left\|\overrightarrow{f(t\omega)}\right\|_q}\right)dt, \tag{141}$$

$\forall\ \omega \in S^{N-1}$.

Here we have $R_1 \leq r \leq R_2$, and $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$, and $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$, also $r^{N-1}r^{1-N} = 1$. Thus by (141), we have

$$\int_{R_1}^{R_2} L_{q*}^-(r\omega)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_2-}^{\gamma;\psi} f(r\omega)}\right|}{L_{q*}^-(r\omega)}\right)r^{N-1}dr \leq \left(\frac{R_2}{R_1}\right)^{N-1} \int_{R_1}^{R_2} W_{q*}^-(r\omega)\Phi\left(\frac{\left|\overrightarrow{f(r\omega)}\right|}{\left\|\overrightarrow{f(r\omega)}\right\|_q}\right)r^{N-1}dr, \tag{142}$$

$\forall\ \omega \in S^{N-1}$.

Therefore it holds

$$\int_{S^{N-1}}\left(\int_{R_1}^{R_2} L_{q*}^{-}(r\omega)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_2-}^{\gamma;\psi} f(r\omega)}\right|}{L_{q*}^{-}(r\omega)}\right) r^{N-1}dr\right)d\omega \leq \left(\frac{R_2}{R_1}\right)^{N-1}\int_{S^{N-1}}\left(\int_{R_1}^{R_2} W_{q*}^{-}(r\omega)\Phi\left(\frac{\left|\overrightarrow{f(r\omega)}\right|}{\left\|\overrightarrow{f(r\omega)}\right\|_q}\right) r^{N-1}dr\right)d\omega. \quad (143)$$

Using Theorem 39 we derive:

Theorem 44 All as in Remark 43. Then

$$\int_A L_{q*}^{-}(x)\Phi\left(\frac{\left|\overrightarrow{e_{\rho,\mu,w,R_2-}^{\gamma;\psi} f(x)}\right|}{L_{q*}^{-}(x)}\right)dx \leq \left(\frac{R_2}{R_1}\right)^{N-1}\int_A W_{q*}^{-}(x)\Phi\left(\frac{\left|\overrightarrow{f(x)}\right|}{\left\|\overrightarrow{f(x)}\right\|_q}\right)dx, \quad (144)$$

where $\overrightarrow{e_{\rho,\mu,w,R_2-}^{\gamma;\psi} f(x)} = \left(\left(e_{\rho,\mu,w,R_2-}^{\gamma;\psi} f_1\right)(x),\ldots,\left(e_{\rho,\mu,w,R_2-}^{\gamma;\psi} f_n\right)(x)\right)$ and coordinates are assumed to be continuous functions on $\overline{A}$.

We need

Definition 45 Let $\rho,\mu,w>0,\ \gamma<0$, $N=\lceil\mu\rceil$, $\mu\notin\mathbb{N}$; $f\in C^N\left(\overline{A}\right)$ and $\psi\in C^N([R_1,R_2]),\ \psi'(r)\neq 0$, $\forall$ $r\in[R_1,R_2]$, and ψ is increasing. We define the ψ-Prabhakar-Caputo radial left and right fractional derivatives of order μ as follows ($x\in\overline{A}$; $x=r\omega$, $r\in[R_1,R_2]$, $\omega\in S^{N-1}$)

$$\left({}_R^C D_{\rho,\mu,w,R_1+}^{\gamma;\psi} f\right)(x) = \left({}_R^C D_{\rho,\mu,w,R_1+}^{\gamma;\psi} f\right)(r\omega) :=$$

$$\int_{R_1}^{r}\psi'(t)(\psi(r)-\psi(t))^{N-\mu-1}E_{\rho,N-\mu}^{-\gamma}\left[w(\psi(r)-\psi(t))^{\rho}\right]\left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^N f(t\omega)dt \quad (145)$$

$$\overset{(125)}{=}\left(e_{\rho,N-\mu,w,R_1+}^{-\gamma;\psi} f_\psi^{[N]}\right)(x),$$

where

$$f_\psi^{[N]}(x) = f_\psi^{[N]}(r\omega) := \left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^N f(r\omega), \quad (146)$$

is the N th order ψ-radial derivative of f,

and

$$\left({}_R^C D_{\rho,\mu,w,R_2-}^{\gamma;\psi} f\right)(x) = \left({}_R^C D_{\rho,\mu,w,R_2-}^{\gamma;\psi} f\right)(r\omega) :=$$

$$(-1)^N\int_{r}^{R_2}\psi'(t)(\psi(t)-\psi(r))^{N-\mu-1}E_{\rho,N-\mu}^{-\gamma}\left[w(\psi(t)-\psi(r))^{\rho}\right] \quad (147)$$

$$\left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^N f(t\omega)dt \overset{(126)}{=} (-1)^N\left(e_{\rho,N-\mu,w,R_2-}^{-\gamma;\psi} f_\psi^{[N]}\right)(x),$$

$\forall$ $x\in\overline{A}$.

In this work we assume that $\left({}^{C}_{R}D^{\gamma;\psi}_{\rho,\mu,w,R_1+}f\right)$ and $\left({}^{C}_{R}D^{\gamma;\psi}_{\rho,\mu,w,R_2-}f\right)$ are continuous functions over $\overline{A}$.

We make

Remark 46 Let $\rho,\mu,w>0,\ \gamma<0$, $N=\lceil\mu\rceil$, $\mu\notin\mathbb{N}$; $f_i\in C^N\left(\overline{A}\right)$, $i=1,\dots,n$, and $\vec{f}=\left(f_1,\dots,f_n\right)$, and $\psi\in C^N\left(\left[R_1,R_2\right]\right)$, $\psi'(r)\neq 0$, $\forall\ r\in\left[R_1,R_2\right]$, and ψ is increasing. We follow Definition 45 and we set:

$$\left\|\overrightarrow{f_{\psi}^{[N]}(y)}\right\|_{\infty}:=\max\left\{\left|f_{1\psi}^{[N]}(y)\right|,\dots,\left|f_{n\psi}^{[N]}(y)\right|\right\},$$
and
$$\left\|\overrightarrow{f_{\psi}^{[N]}(y)}\right\|_{q}:=\left(\sum_{i=1}^{n}\left|f_{i\psi}^{[N]}(y)\right|^q\right)^{\frac{1}{q}},\ q\geq 1;\ y\in\overline{A}. \tag{148}$$

One can write that

$$\left\|\overrightarrow{f_{\psi}^{[N]}(y)}\right\|_{q}=\left\|\overrightarrow{f_{\psi}^{[N]}(t\omega)}\right\|_{q},\ 1\leq q\leq\infty, \tag{149}$$

where $t\in\left[R_1,R_2\right]$, $\omega\in S^{N-1}$; $y=t\omega$.

Notice that $\left\|\overrightarrow{f_{\psi}^{[N]}(y)}\right\|_{q}\in C\left(\overline{A}\right)$, $1\leq q\leq\infty$.

We assume that $\left\|\overrightarrow{f_{\psi}^{[N]}(y)}\right\|_{q}>0$ on $\overline{A}$, $1\leq q\leq\infty$ fixed.

Consider the kernel

$${}^{C}k^{+}(r,t):=k(r,t):=\chi_{(R_1,r]}(t)\psi'(t)\left(\psi(r)-\psi(t)\right)^{N-\mu-1}E^{-\gamma}_{\rho,N-\mu}\left[w\left(\psi(r)-\psi(t)\right)^{\rho}\right]. \tag{150}$$

Let

$${}^{C}L_q^{+}(x)={}^{C}L_q^{+}(r\omega)=\int_{R_1}^{R_2}{}^{C}k^{+}(r,t)\left\|\overrightarrow{f_{\psi}^{[N]}(t\omega)}\right\|_{q}dt, \tag{151}$$

$x=r\omega\in\overline{A}$, $1\leq q\leq\infty$ fixed; $r\in\left[R_1,R_2\right]$, $\omega\in S^{N-1}$.

We have that ${}^{C}L_q^{+}(r\omega)>0$ for $r\in(R_1,R_2]$, $\forall\ \omega\in S^{N-1}$.

Here we choose the weight $u(x)=u(r\omega)={}^{C}L_q^{+}(r\omega)$.

Consider the function

$${}^{C}W_q^{+}(y)={}^{C}W_q^{+}(t\omega)=\left\|\overrightarrow{f_{\psi}^{[N]}(t\omega)}\right\|_{q}\left(\int_{R_1}^{R_2}{}^{C}k^{+}(r,t)dr\right)<\infty, \tag{152}$$

$\forall\ t\in\left[R_1,R_2\right]$, $\omega\in S^{N-1}$; and ${}^{C}W_q^{+}(t\omega)$ is integrable over $\left[R_1,R_2\right]$, $\forall\ \omega\in S^{N-1}$.

Here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (145) follows:

Theorem 47 All as in Remark 46. Then

$$\int_A {}^C L_q^+(x) \Phi \left(\frac{\left| \overrightarrow{\left({}_R^C D_{\rho,\mu,w,R_1+}^{\gamma;\psi} f \right)(x)} \right|}{{}^C L_q^+(x)} \right) dx \leq \left(\frac{R_2}{R_1} \right)^{N-1} \int_A {}^C W_q^+(x) \Phi \left(\frac{\left| \overrightarrow{f_\psi^{[N]}(x)} \right|}{\left\| \overrightarrow{f_\psi^{[N]}(x)} \right\|_q} \right) dx, \tag{153}$$

where $\overrightarrow{\left({}_R^C D_{\rho,\mu,w,R_1+}^{\gamma;\psi} f \right)(x)} = \left(\left({}_R^C D_{\rho,\mu,w,R_1+}^{\gamma;\psi} f_1 \right)(x), \ldots, \left({}_R^C D_{\rho,\mu,w,R_1+}^{\gamma;\psi} f_n \right)(x) \right)$ and the coordinates are assumed to be continuous on $\overline{A}$.

We make

Remark 48 Let $\rho, \mu, w > 0$, $\gamma < 0$, $N = \lceil \mu \rceil$, $\mu \notin \mathbb{N}$; $f_i \in C^N(\overline{A})$, $i = 1, \ldots, n$, and $\vec{f} = (f_1, \ldots, f_n)$, and $\psi \in C^N([R_1, R_2])$, $\psi'(r) \neq 0$, $\forall$ $r \in [R_1, R_2]$, and ψ is increasing. We follow Definition 45 and we set:

$$\left\| \overrightarrow{f_\psi^{[N]}(y)} \right\|_\infty := \max \left\{ \left| f_{1\psi}^{[N]}(y) \right|, \ldots, \left| f_{n\psi}^{[N]}(y) \right| \right\},$$
$$\text{and}$$
$$\left\| \overrightarrow{f_\psi^{[N]}(y)} \right\|_q := \left(\sum_{i=1}^n \left| f_{i\psi}^{[N]}(y) \right|^q \right)^{\frac{1}{q}}, \; q \geq 1; \; y \in \overline{A}. \tag{154}$$

One can write that

$$\left\| \overrightarrow{f_\psi^{[N]}(y)} \right\|_q = \left\| \overrightarrow{f_\psi^{[N]}(t\omega)} \right\|_q, \; 1 \leq q \leq \infty, \tag{155}$$

where $t \in [R_1, R_2]$, $\omega \in S^{N-1}$; $y = t\omega$.

Notice that $\left\| \overrightarrow{f_\psi^{[N]}(y)} \right\|_q \in C(\overline{A})$, $1 \leq q \leq \infty$.

We assume that $\left\| \overrightarrow{f_\psi^{[N]}(y)} \right\|_q > 0$ on $\overline{A}$, $1 \leq q \leq \infty$ fixed.

Consider the kernel

$${}^C k^-(r,t) := k(r,t) := \chi_{r,R_2)}(t) \psi'(t) (\psi(t) - \psi(r))^{N-\mu-1} E_{\rho,N-\mu}^{-\gamma} \left[w (\psi(t) - \psi(r))^\rho \right]. \tag{156}$$

Let

$${}^C L_q^-(x) = {}^C L_q^-(r\omega) = \int_{R_1}^{R_2} {}^C k^-(r,t) \left\| \overrightarrow{f_\psi^{[N]}(t\omega)} \right\|_q dt, \tag{157}$$

$x = r\omega \in \overline{A}$, $1 \leq q \leq \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We have that ${}^{C}L_q^{-}(r\omega)>0$ for $r\in(R_1,R_2]$, $\forall\ \omega\in S^{N-1}$.

Here we choose the weight $u(x)=u(r\omega)={}^{C}L_q^{-}(r\omega)$.

Consider the function

$$ {}^{C}W_q^{-}(y)={}^{C}W_q^{-}(t\omega)=\left\|\overrightarrow{f_{\psi}^{[N]}(t\omega)}\right\|_q\left(\int_{R_1}^{R_2}{}^{C}k^{-}(r,t)dr\right)<\infty, \tag{158}$$

$\forall\ t\in[R_1,R_2]$, $\omega\in S^{N-1}$; and ${}^{C}W_q^{-}(t\omega)$ is integrable over $[R_1,R_2]$, $\forall\ \omega\in S^{N-1}$.

Here $\Phi:\mathbb{R}_+^n\to\mathbb{R}$ is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (147) follows:

Theorem 49 All as in Remark 48. Then

$$\int_A {}^{C}L_q^{-}(x)\Phi\left(\frac{\left|\overrightarrow{\left({}_{R}^{C}D_{\rho,\mu,w,R_2-}^{\gamma;\psi}f\right)(x)}\right|}{{}^{C}L_q^{-}(x)}\right)dx\le\left(\frac{R_2}{R_1}\right)^{N-1}\int_A {}^{C}W_q^{-}(x)\Phi\left(\frac{\left|\overrightarrow{f_{\psi}^{[N]}(x)}\right|}{\left\|\overrightarrow{f_{\psi}^{[N]}(x)}\right\|_q}\right)dx, \tag{159}$$

where $\overrightarrow{\left({}_{R}^{C}D_{\rho,\mu,w,R_2-}^{\gamma;\psi}f\right)(x)}=\left(\left({}_{R}^{C}D_{\rho,\mu,w,R_2-}^{\gamma;\psi}f_1\right)(x),\ldots,\left({}_{R}^{C}D_{\rho,\mu,w,R_2-}^{\gamma;\psi}f_n\right)(x)\right)$ and the coordinates are assumed to be continuous on $\overline{A}$.

We need

Definition 50 Let $\rho,\mu,w>0,\ \gamma<0$, $N=\lceil\mu\rceil$, $\mu\notin\mathbb{N}$; $f\in C(\overline{A})$ and $\psi\in C^N([R_1,R_2]),\ \psi'(r)\neq 0$, $\forall$ $r\in[R_1,R_2]$, and ψ is increasing. The ψ-Prabhakar-Riemann Liouville left and right radial fractional derivatives of order μ are defined as follows (see also Definition 40)

$$\left({}_{R}^{RL}D_{\rho,\mu,w,R_1+}^{\gamma;\psi}f\right)(x)=\left({}_{R}^{RL}D_{\rho,\mu,w,R_1+}^{\gamma;\psi}f\right)(r\omega):=\left(\frac{1}{\psi'(r)}\frac{d}{dr}\right)^N\left(e_{\rho,N-\mu,w,R_1+}^{-\gamma;\psi}f\right)(x), \tag{160}$$

and

$$\left({}_{R}^{RL}D_{\rho,\mu,w,R_2-}^{\gamma;\psi}f\right)(x)=\left({}_{R}^{RL}D_{\rho,\mu,w,R_2-}^{\gamma;\psi}f\right)(r\omega):=\left(-\frac{1}{\psi'(r)}\frac{d}{dr}\right)^N\left(e_{\rho,N-\mu,w,R_2-}^{-\gamma;\psi}f\right)(x), \tag{161}$$

$\forall\ x\in\overline{A}$; where $x=r\omega$, $r\in[R_1,R_2]$, $\omega\in S^{N-1}$.

In this work we assume that $\left({}_{R}^{RL}D_{\rho,\mu,w,R_1+}^{\gamma;\psi}f\right)$, $\left({}_{R}^{RL}D_{\rho,\mu,w,R_2-}^{\gamma;\psi}f\right)\in C(\overline{A})$.

Next we define the ψ-Hilfer-Prabhakar left and right radial fractional derivatives of order μ and type $\beta\in[0,1]$, as follows ($\xi:=\mu+\beta(N-\mu)$, see also Definition 40):

$$\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi}f\right)(x)=\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi}f\right)(r\omega):=e_{\rho,\xi-\mu,w,R_1+}^{-\gamma\beta;\psi}\left({}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f\right)(x), \tag{162}$$

and

$$\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi}f\right)(x)=\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi}f\right)(r\omega):=e_{\rho,\xi-\mu,w,R_2-}^{-\gamma\beta;\psi}\left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi}f\right)(x), \tag{163}$$

$\forall\ x\in\overline{A}$; where $x=r\omega$, $r\in[R_1,R_2]$, $\omega\in S^{N-1}$.

In this work we assume that $\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi}f\right)$, $\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi}f\right)\in C\left(\overline{A}\right)$.

We make

Remark 51 Let $\rho,\mu,w>0,\ \gamma<0$, $N=\lceil\mu\rceil$, $\mu\notin\mathbb{N};\ 0\le\beta\le 1$, $\xi=\mu+\beta(N-\mu)$, $f_i\in C\left(\overline{A}\right)$, $i=1,\dots,n,$ and $\psi\in C^N([R_1,R_2])$, $\psi'(r)\neq 0$, $\forall\ r\in[R_1,R_2]$, and ψ is increasing. We follow Definition 50, especially (162) and we set:

$$\begin{gathered}\left\|\left(\overrightarrow{{}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f}\right)(y)\right\|_\infty:=\\ \max\left\{\left|\left({}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f_1\right)(y)\right|,\dots,\left|\left({}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f_n\right)(y)\right|\right\},\\ \text{and}\\ \left\|\left(\overrightarrow{{}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f}\right)(y)\right\|_q:=\left(\sum_{i=1}^n\left|\left({}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f_i\right)(y)\right|^q\right)^{\frac{1}{q}},\ q\ge 1;\ y\in\overline{A}.\end{gathered} \tag{164}$$

One can write that

$$\left\|\left(\overrightarrow{{}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f}\right)(y)\right\|_q=\left\|\left(\overrightarrow{{}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f}\right)(t\omega)\right\|_q,\ 1\le q\le\infty, \tag{165}$$

where $t\in[R_1,R_2]$, $\omega\in S^{N-1}$; $y=t\omega$.

Notice that $\left\|\left(\overrightarrow{{}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f}\right)(y)\right\|_q\in C\left(\overline{A}\right)$, $1\le q\le\infty$.

We assume that $\left\|\left(\overrightarrow{{}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f}\right)(y)\right\|_q>0$ on $\overline{A}$, $1\le q\le\infty$ fixed.

Consider the kernel

$${}^{P}k^{+}(r,t):=k(r,t):=\quad\chi_{(R_1,r]}(t)\psi'(t)(\psi(r)-\psi(t))^{\xi-\mu-1}E_{\rho,\xi-\mu}^{-\gamma\beta}\left[w(\psi(r)-\psi(t))^{\rho}\right]. \tag{166}$$

Let

$${}^{P}L_q^{+}(x)={}^{P}L_q^{+}(r\omega)=\int_{R_1}^{R_2}{}^{P}k^{+}(r,t)\left\|\left(\overrightarrow{{}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi}f}\right)(t\omega)\right\|_q dt, \tag{167}$$

$x = r\omega \in \overline{A}$, $1 \leq q \leq \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We have that ${}^{P}L_q^{+}(r\omega) > 0$ for $r \in (R_1, R_2]$, $\forall\ \omega \in S^{N-1}$.

Here we choose the weight $u(x) = u(r\omega) = {}^{P}L_q^{+}(r\omega)$.

Consider the function

$$ {}^{P}W_q^{+}(y) = {}^{P}W_q^{+}(t\omega) = \left\| \left({}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi} \overrightarrow{f} \right)(t\omega) \right\|_q \left(\int_{R_1}^{R_2} {}^{P}k^{+}(r,t)\,dr \right) < \infty, \tag{168} $$

$\forall\ t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and ${}^{P}W_q^{+}(t\omega)$ is integrable over $[R_1, R_2]$, $\forall\ \omega \in S^{N-1}$.

Here $\Phi : \mathbb{R}_{+}^{n} \to \mathbb{R}$ is a convex and increasing per coordinate function.

A direct application of Theorem 42, along with (162) follows:

Theorem 52 All as in Remark 51. Then

$$ \int_A {}^{P}L_q^{+}(x)\Phi\left(\frac{\left| \left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi} \overrightarrow{f} \right)(x) \right|}{{}^{P}L_q^{+}(x)} \right) dx \leq \left(\frac{R_2}{R_1} \right)^{N-1} \int_A {}^{P}W_q^{+}(x)\Phi\left(\frac{\left| \left({}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi} \overrightarrow{f} \right)(x) \right|}{\left\| \left({}_{R}^{RL}D_{\rho,\xi,w,R_1+}^{\gamma(1-\beta);\psi} \overrightarrow{f} \right)(x) \right\|_q} \right) dx, \tag{169} $$

where $\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi} \overrightarrow{f} \right)(x) = \left(\left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi} f_1 \right)(x), \ldots, \left({}_{R}^{H}\mathsf{D}_{\rho,\mu,w,R_1+}^{\gamma,\beta;\psi} f_n \right)(x) \right)$ and the coordinates are assumed to be continuous on $\overline{A}$.

We make

Remark 53 Let $\rho, \mu, w > 0,\ \gamma < 0$, $N = \lceil \mu \rceil$, $\mu \notin \mathbb{N};\ 0 \leq \beta \leq 1$, $\xi = \mu + \beta(N - \mu)$, $f_i \in C(\overline{A})$, $i = 1,\ldots,n,$ and $\psi \in C^N([R_1, R_2])$, $\psi'(r) \neq 0$, $\forall\ r \in [R_1, R_2]$, and ψ is increasing. We follow Definition 50, especially (163) and we set:

$$ \begin{gathered} \left\| \left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} \overrightarrow{f} \right)(y) \right\|_\infty := \\ \max\left\{ \left| \left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f_1 \right)(y) \right|, \ldots, \left| \left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f_n \right)(y) \right| \right\}, \\ \text{and} \\ \left\| \left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} \overrightarrow{f} \right)(y) \right\|_q := \left(\sum_{i=1}^{n} \left| \left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f_i \right)(y) \right|^q \right)^{\frac{1}{q}},\ q \geq 1;\ y \in \overline{A}. \end{gathered} \tag{170} $$

One can write that

$$ \left\| \left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} \overrightarrow{f} \right)(y) \right\|_q = \left\| \left({}_{R}^{RL}D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} \overrightarrow{f} \right)(t\omega) \right\|_q,\ 1 \leq q \leq \infty, \tag{171} $$

where $t \in [R_1, R_2]$, $\omega \in S^{N-1}$; $y = t\omega$.

Notice that $\left\| \left(\overrightarrow{{}_R^{RL} D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f} \right)(y) \right\|_q \in C(\overline{A})$, $1 \le q \le \infty$.

We assume that $\left\| \left(\overrightarrow{{}_R^{RL} D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f} \right)(y) \right\|_q > 0$ on $\overline{A}$, $1 \le q \le \infty$ fixed.

Consider the kernel

$$ {}^P k^-(r,t) := k(r,t) := \quad \chi_{[r,R_2)}(t)\psi'(t)(\psi(t)-\psi(r))^{\xi-\mu-1} E_{\rho,\xi-\mu}^{-\gamma\beta}\left[w(\psi(t)-\psi(r))^{\rho}\right]. \tag{172} $$

Let

$$ {}^P L_q^-(x) = {}^P L_q^-(r\omega) = \int_{R_1}^{R_2} {}^P k^-(r,t) \left\| \left(\overrightarrow{{}_R^{RL} D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f} \right)(t\omega) \right\|_q dt, \tag{173} $$

$x = r\omega \in \overline{A}$, $1 \le q \le \infty$ fixed; $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We have that ${}^P L_q^-(r\omega) > 0$ for $r \in (R_1, R_2]$, $\forall\ \omega \in S^{N-1}$.

Here we choose the weight $u(x) = u(r\omega) = {}^P L_q^-(r\omega)$.

Consider the function

$$ {}^P W_q^-(y) = {}^P W_q^-(t\omega) = \left\| \left(\overrightarrow{{}_R^{RL} D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f} \right)(t\omega) \right\|_q \left(\int_{R_1}^{R_2} {}^P k^-(r,t) dr \right) < \infty, \tag{174} $$

$\forall\ t \in [R_1, R_2]$, $\omega \in S^{N-1}$; and ${}^P W_q^-(t\omega)$ is integrable over $[R_1, R_2]$, $\forall\ \omega \in S^{N-1}$.

Here $\Phi : \mathbb{R}_+^n \to \mathbb{R}$ is a convex and increasing per coordinate function.

A direct application of Theorem 44, along with (163) follows:

Theorem 54 All as in Remark 53. Then

$$ \int_A {}^P L_q^-(x) \Phi\left(\frac{\left(\overrightarrow{{}_R^H \mathsf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi} f} \right)(x)}{{}^P L_q^-(x)} \right) dx \le \quad \left(\frac{R_2}{R_1} \right)^{N-1} \int_A {}^P W_q^-(x) \Phi\left(\frac{\left| \left(\overrightarrow{{}_R^{RL} D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f} \right)(x) \right|}{\left\| \left(\overrightarrow{{}_R^{RL} D_{\rho,\xi,w,R_2-}^{\gamma(1-\beta);\psi} f} \right)(x) \right\|_q} \right) dx, \tag{175} $$

where $\left(\overrightarrow{{}_R^H \mathsf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi} f} \right)(x) = \left(\left({}_R^H \mathsf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi} f_1 \right)(x), \ldots, \left({}_R^H \mathsf{D}_{\rho,\mu,w,R_2-}^{\gamma,\beta;\psi} f_n \right)(x) \right)$ and the coordinates are assumed to be continuous on $\overline{A}$.

We make

Remark 55 Let $f_{ji} \in C([a,b])$, $j = 1,2; i = 1,\ldots,m; \psi \in C^1([a,b])$ which is increasing. Let also $\rho_i, \mu_i, \gamma_i, \omega_i > 0$

and $\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_{ji}\right)(x)$, $x\in[a,b]$ as in (2). We assume here that $0<f_{2i}(y)<\infty$ on $[a,b]$, $i=1,...,m.$

Here we consider the kernel

$$k_i^+(x,y):=k_i(x,y)=\begin{cases}\psi'(y)(\psi(x)-\psi(y))^{\mu_i-1}E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(x)-\psi(y))^{\rho_i}\right], a<y\le x,\\ \\ 0,\ x<y<b,\end{cases} \tag{176}$$

$i=1,...,m.$

Choose weight $u(x)\ge 0$, so that

$$\psi_i(y):=f_{2i}(y)\int_y^b u(x)\frac{k_i^+(x,y)}{\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_{2i}\right)(x)}dx<\infty, \tag{177}$$

a.e. on $[a,b]$, and that ψ_i is integrable on $[a,b]$, $i=1,...,m.$

Theorem 9 immediately implies:

Theorem 56 All as in Remark 55. Let $p_i>1:\sum_{i=1}^m\frac{1}{p_i}=1$. Let the functions $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,...,m,$ be convex and increasing. Then

$$\int_a^b u(x)\prod_{i=1}^m\Phi_i\left(\frac{\left|\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_{1i}\right)(x)\right|}{\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+}f_{2i}\right)(x)}\right)dx\le\prod_{i=1}^m\left(\int_a^b\psi_i(y)\Phi_i\left(\frac{|f_{1i}(y)|}{f_{2i}(y)}\right)^{p_i}dy\right)^{\frac{1}{p_i}}. \tag{178}$$

We make

Remark 57 Let $f_{ji}\in C([a,b])$, $j=1,2;\ i=1,...,m;\ \psi\in C^1([a,b])$ which is increasing. Let also $\rho_i,\mu_i,\gamma_i,\omega_i>0$ and $\left(e^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_{ji}\right)(x)$, $x\in[a,b]$ as in (3). We assume here that $0<f_{2i}(y)<\infty$ on $[a,b]$, $i=1,...,m.$

Here we consider the kernel

$$k_i^-(x,y):=k_i(x,y)=\begin{cases}\psi'(y)(\psi(y)-\psi(x))^{\mu_i-1}E^{\gamma_i}_{\rho_i,\mu_i}\left[\omega_i(\psi(y)-\psi(x))^{\rho_i}\right], x\le y<b,\\ \\ 0,\ a<y<x,\end{cases} \tag{179}$$

$i=1,...,m.$

Choose weight $u(x)\ge 0$, so that

$$\overline{\psi_i}(y) := f_{2i}(y) \int_a^y u(x) \frac{k_i^-(x,y)}{\left(e_{\rho_i,\mu_i,\omega_i,b-}^{\gamma_i;\psi} f_{2i}\right)(x)} dx < \infty, \tag{180}$$

a.e. on $[a,b]$, and that $\overline{\psi_i}$ is integrable on $[a,b]$, $i = 1,\dots,m.$

Theorem 9 immediately implies:

Theorem 58 All as in Remark 57. Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\dots,m,$ be convex and increasing. Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left| \left(e_{\rho_i,\mu_i,\omega_i,b-}^{\gamma_i;\psi} f_{1i}\right)(x) \right|}{\left(e_{\rho_i,\mu_i,\omega_i,b-}^{\gamma_i;\psi} f_{2i}\right)(x)} \right) dx \leq \prod_{i=1}^m \left(\int_a^b \overline{\psi_i}(y) \Phi_i \left(\frac{|f_{1i}(y)|}{f_{2i}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}}. \tag{181}$$

We make

Remark 59 Let $j = 1,2; i = 1,\dots,n;$ $\rho_i, \mu_i, \omega_i > 0,\ \gamma_i < 0,\ N_i = \lceil \mu_i \rceil,$ $\mu_i \notin \mathbb{N}; \theta := \max(N_1,\dots,N_m),$ $\psi \in C^\theta([a,b]),\ \psi'(x) \neq 0$ over $[a,b]$, ψ is increasing; $f_{ji} \in C^{N_i}([a,b])$ and $f_{ji\psi}^{[N_i]}(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{N_i} f_{ji}(x)$, $\forall$ $x \in [a,b]$. Here

$$\left({}^C D_{\rho_i,\mu_i,\omega_i,a+}^{\gamma_i;\psi} f_{ji}\right)(x) \overset{(6)}{=} \left(e_{\rho_i,N_i-\mu_i,\omega_i,a+}^{-\gamma_i;\psi} f_{ji\psi}^{[N_i]}\right)(x), \tag{182}$$

$\forall$ $x \in [a,b]$, $j = 1,2; i = 1,\dots,m.$

We assume that $0 < f_{2i\psi}^{[N_i]}(y) < \infty$ on $[a,b]$, $i = 1,\dots,m.$

Here we consider the kernel

$${}^C k_i^+(x,y) := k_i(x,y) = \begin{cases} \psi'(y)(\psi(x)-\psi(y))^{N_i-\mu_i-1} E_{\rho_i,N_i-\mu_i}^{-\gamma_i}\left[\omega_i(\psi(x)-\psi(y))^{\rho_i}\right], a < y \leq x, \\ 0, x < y < b, \end{cases} \tag{183}$$

$i = 1,\dots,m.$

Choose weight $u \geq 0,$ so that

$${}^C\psi_i(y) := f_{2i\psi}^{[N_i]}(y) \int_y^b u(x) \frac{{}^C k_i^+(x,y)}{\left({}^C D_{\rho_i,\mu_i,\omega_i,a+}^{\gamma_i;\psi} f_{2i}\right)(x)} dx < \infty, \tag{184}$$

a.e. on $[a,b]$, and that ${}^C\psi_i$ is integrable on $[a,b]$, $i = 1,\dots,m.$

Theorem 56 immediately produces:

Theorem 60 All as in Remark 59. Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,...,m$, be convex and increasing. Then

$$\int_a^b u(x) \prod_{i=1}^{m} \Phi_i \left(\frac{\left| \left({}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_{1i} \right)(x) \right|}{\left({}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_{2i} \right)(x)} \right) dx \leq \prod_{i=1}^{m} \left(\int_a^b {}^{C}\psi_i(y) \Phi_i \left(\frac{\left| f^{[N_i]}_{1i\psi}(y) \right|}{f^{[N_i]}_{2i\psi}(y)} \right)^{p_i} dy \right)^{\frac{1}{p_i}}. \tag{185}$$

We make

Remark 61 Let $j = 1,2; i = 1,...,n;$ $\rho_i, \mu_i, \omega_i > 0$, $\gamma_i < 0$, $N_i = \lceil \mu_i \rceil$, $\mu_i \notin \mathbb{N}$; $\theta := \max(N_1,..., N_m)$, $\psi \in C^{\theta}([a,b])$, $\psi'(x) \neq 0$ over $[a,b]$, ψ is increasing; $f_{ji} \in C^{N_i}([a,b])$ and $f^{[N_i]}_{ji\psi}(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^{N_i} f_{ji}(x)$, $\forall$ $x \in [a,b]$. Here

$$\left({}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-} f_{ji} \right)(x) \overset{(7)}{=} (-1)^{N_i} \left(e^{-\gamma_i;\psi}_{\rho_i,N_i-\mu_i,\omega_i,b-} f^{[N_i]}_{ji\psi} \right)(x), \tag{186}$$

$\forall$ $x \in [a,b]$, $j = 1,2; i = 1,...,m.$

We assume that $0 < f^{[N_i]}_{2i\psi}(y) < \infty$ on $[a,b]$, $i = 1,...,m.$

Here we consider the kernel

$${}^{C}k_i^-(x,y) := k_i(x,y) = \begin{cases} \psi'(y)(\psi(y) - \psi(x))^{N_i-\mu_i-1} E^{-\gamma_i}_{\rho_i,N_i-\mu_i} \left[\omega_i (\psi(y) - \psi(x))^{\rho_i} \right], x \leq y < b, \\ 0, a < y < x, \end{cases} \tag{187}$$

$i = 1,...,m.$

Choose weight $u \geq 0$, so that

$${}^{C}\overline{\psi_i}(y) := f^{[N_i]}_{2i\psi}(y) \int_a^y u(x) \frac{{}^{C}k_i^-(x,y)}{\left({}^{C}D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-} f_{2i} \right)(x)} dx < \infty, \tag{188}$$

a.e. on $[a,b]$, and that ${}^{C}\overline{\psi_i}$ is integrable on $[a,b]$, $i = 1,...,m.$

Theorem 58 immediately produces:

Theorem 62 All as in Remark 61. Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,...,m$, be convex and increasing. Then

$$\int_a^b u(x)\prod_{i=1}^m \Phi_i\left(\frac{\left|\left({}^C D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-} f_{1i}\right)(x)\right|}{\left({}^C D^{\gamma_i;\psi}_{\rho_i,\mu_i,\omega_i,b-} f_{2i}\right)(x)}\right)dx \leq \prod_{i=1}^m\left(\int_a^b {}^C\overline{\psi}_i(y)\Phi_i\left(\frac{\left|f_{1i\psi}^{[N_i]}(y)\right|}{f_{2i\psi}^{[N_i]}(y)}\right)^{p_i} dy\right)^{\frac{1}{p_i}}. \tag{189}$$

We make

Remark 63 Let $j=1,2;\, i=1,\dots,m;\ \rho_i,\mu_i,\omega_i>0,\ \gamma_i<0,\ N_i=\lceil\mu_i\rceil,\quad \mu_i\notin \mathbb{N};\ \theta:=\max(N_1,\dots,N_m)$, $\psi\in C^\theta([a,b]),\ \psi'(x)\neq 0$ over $[a,b]$, ψ is increasing; $f_{ji}\in C([a,b])$. Let $0\le\beta_i\le 1$ and $\xi_i=\mu_i+\beta_i(N_i-\mu_i)$, $i=1,\dots,m$. We assume that ${}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_{ji}\in C([a,b])$ and $0<{}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_{2i}(y)<\infty$ on $[a,b]$, $i=1,\dots,m$. Here we have

$$\left({}^H\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_{ji}\right)(x) \overset{(15)}{=} e^{-\gamma_i\beta_i;\psi}_{\rho_i,\xi_i-\mu_i,\omega_i,a+}\ {}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_{ji}(x), \tag{190}$$

$\forall\ x\in[a,b]$, $j=1,2;\ i=1,\dots,m.$

Here we consider the kernel

$${}^P k_i^+(x,y):=k_i(x,y)=\begin{cases}\psi'(y)(\psi(x)-\psi(y))^{\xi_i-\mu_i-1}E^{-\gamma_i\beta_i}_{\rho_i,\xi_i-\mu_i}\left[\omega_i(\psi(x)-\psi(y))^{\rho_i}\right], a<y\le x,\\ 0,\ x<y<b,\end{cases} \tag{191}$$

$i=1,\dots,m.$

Choose weight $u\ge 0$, so that

$${}^P\psi_i(y):=\left({}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_{2i}(y)\right)\int_y^b \frac{u(x)\,{}^P k_i^+(x,y)}{\left({}^H\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_{2i}\right)(x)}dx<\infty, \tag{192}$$

a.e. on $[a,b]$, and that ${}^P\psi_i$ is integrable on $[a,b]$, $i=1,\dots,m.$

Theorem 56 immediately produces:

Theorem 64 All as in Remark 63. Let $p_i>1:\sum_{i=1}^m\frac{1}{p_i}=1$. Let the functions $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,\dots,m,$ be convex and increasing. Then

$$\int_a^b u(x)\prod_{i=1}^m \Phi_i\left(\frac{\left|\left({}^H\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_{1i}\right)(x)\right|}{\left({}^H\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,a+} f_{2i}\right)(x)}\right)dx \leq \prod_{i=1}^m\left(\int_a^b {}^P\psi_i(y)\Phi_i\left(\frac{\left|\left({}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_{1i}\right)(y)\right|}{\left({}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,a+} f_{2i}\right)(y)}\right)^{p_i} dy\right)^{\frac{1}{p_i}}. \tag{193}$$

We make

Remark 65 Let $j=1,2;\ i=1,...,m;$ $\rho_i,\mu_i,\omega_i>0,\ \gamma_i<0,\ N_i=\lceil \mu_i \rceil,$ $\mu_i\notin \mathbb{N};\ \theta:=\max(N_1,...,N_m),$ $\psi\in C^{\theta}([a,b]),\ \psi'(x)\neq 0$ over $[a,b],\ \psi$ is increasing; $f_{ji}\in C([a,b])$. Let $0\leq \beta_i\leq 1$ and $\xi_i=\mu_i+\beta_i(N_i-\mu_i)$, $i=1,...,m$. We assume that ${}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-}f_{ji}\in C([a,b])$ and $0<{}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-}f_{2i}(y)<\infty$ on $[a,b]$, $i=1,...,m$. Here we have

$$\left({}^{H}\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_{ji}\right)(x)\overset{(16)}{=}e^{-\gamma_i\beta_i;\psi}_{\rho_i,\xi_i-\mu_i,\omega_i,b-}\ {}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-}f_{ji}(x), \tag{194}$$

$\forall\ x\in[a,b]$, $j=1,2;\ i=1,...,m.$

Here we consider the kernel

$$ {}^{P}k_i^{-}(x,y):=k_i(x,y)=\begin{cases}\psi'(y)(\psi(y)-\psi(x))^{\xi_i-\mu_i-1}E^{-\gamma_i\beta_i}_{\rho_i,\xi_i-\mu_i}\left[\omega_i(\psi(y)-\psi(x))^{\rho_i}\right], x\leq y<b,\\ 0,\ a<y<x,\end{cases} \tag{195}$$

$i=1,...,m.$

Choose weight $u\geq 0,$ so that

$${}^{P}\overline{\psi_i}(y):=\left({}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-}f_{2i}(y)\right)\int_a^y\frac{u(x)\,{}^{P}k_i^{-}(x,y)}{\left({}^{H}\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_{2i}\right)(x)}dx<\infty, \tag{196}$$

a.e. on $[a,b]$, and that ${}^{P}\overline{\psi_i}$ is integrable on $[a,b]$, $i=1,...,m.$

Theorem 58 immediately produces:

Theorem 66 All as in Remark 65. Let $p_i>1:\sum_{i=1}^{m}\frac{1}{p_i}=1$. Let the functions $\Phi_i:\mathbb{R}_+\to\mathbb{R}_+$, $i=1,...,m,$ be convex and increasing. Then

$$\int_a^b u(x)\prod_{i=1}^{m}\Phi_i\left(\frac{\left|\left({}^{H}\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_{1i}\right)(x)\right|}{\left({}^{H}\mathsf{D}^{\gamma_i,\beta_i;\psi}_{\rho_i,\mu_i,\omega_i,b-}f_{2i}\right)(x)}\right)dx\leq \prod_{i=1}^{m}\left(\int_a^b {}^{P}\overline{\psi_i}(y)\Phi_i\left(\frac{\left|\left({}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-}f_{1i}\right)(y)\right|}{\left({}^{RL}D^{\gamma_i(1-\beta_i);\psi}_{\rho_i,\xi_i,\omega_i,b-}f_{2i}\right)(y)}\right)^{p_i}dy\right)^{\frac{1}{p_i}}. \tag{197}$$

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