A Numerical Method with Shifted Chebyshev Polynomials for a Set of Variable Order Fractional Partial Differential Equations

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Abstract: In this paper, a high-efficiency numerical algorithm based on shifted Chebyshev polynomials is given to solve a set of variable-order fractional partial differential equations. First, we structure the differential operator matrix of the shifted Chebyshev polynomials. Then, we transform the problem into solving a set of linear algebraic equations to obtain the numerical solution. Moreover, a step of error correction is given. Finally, numerical examples are given to show the effectiveness and practicability of the proposed method.

Keywords: Shifted Chebyshev polynomials, Variable order fractional partial differential equations, Error correction, Variable order differential operator matrix.

1. INTRODUCTION

As an important module of mathematical theory [1], fractional calculus theory has been successfully applied to various fields, such as physics and statistical mechanics [2-5], viscoelastic material [6-10], signal processing, control theory [11-16], and so on. However, more and more researchers have found that the fractional differential operators can be used to solve complex dynamic problems, such as the model of linear and nonlinear viscoelastic oscillators [17, 18], the change of the fractional-order with time [19, 20], the physics experiment of the fractional-order operator [21], and so on. In [22], the fractional derivative is used to model percolation equations, which can describe the time-dependent percolation process more effectively. So far, many scholars have proposed different definetions of variable-order fractional differential operators, including Riemann-Liouvile definition, Caputo definition, Marchaud definition, and Coimbra definition [23-29].

In recent years, more and more numerical methods for fractional partial differential equations have been studied to approximate analytical solutions. Many scholars have proposed numerical methods for various types of fractional differential equations, including the Chebyshev and Legendre polynomials methods [30-35, 40], Wavelet analysis method [36-39], piecewise constant function methods [41, 42], differential transformation method [43], traveling wave transformation method [44], spectral methods [45, 46], Adomian decomposition method [47] etc. Due to the complexity of variable-order fractional partial differential equations are hard to obtain. Most researchers used different methods to solve the variable-order fractional partial differential equations. For example, in [48] the authors proposed the Legendre polynomials method to solve the variable order linear fractional partial differential equation. Shen S [49] and others employed the finite difference method to solve the variable-order fractional convection-diffusion equation with a nonlinear source term on a finite field. A method of accurate spectral collocation to solve one-dimensional and two-dimensional variable-order fractional nonlinear cable equations was proposed in [50].

In this paper, the shifted Chebyshev polynomials are applied to obtain the numerical solutions for a system of variable-order fractional partial differential equations. Indeed, Chebyshev polynomials have a wide variety of applications and play an indispensable role in solving integral equations and differential equations.

The following forms of a system of variable-order fractional partial differential equations are discussed:

$$\begin{cases} \frac{\partial^{\alpha(x)}u(x,t)}{\partial x^{\alpha(x)}} + \frac{\partial u(x,t)}{\partial x} + g_1(u,v) = f_1(x,t), \\ \frac{\partial^{\beta(t)}v(x,t)}{\partial t^{\beta(t)}} + \frac{\partial v(x,t)}{\partial t} + g_2(u,v) = f_2(x,t), \end{cases} \quad (x,t) \in [0,X]\times[0,T], \tag{1}$$

which satisfy the following initial conditions and boundary conditions:

$$\begin{aligned} &u(0,t) = h_1(t), u(x,0) = \varphi_1(x), \\ &v(0,t) = h_2(t), v(x,0) = \varphi_2(x), \end{aligned} \tag{2}$$

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where $0<\alpha(x)<1, 0<\beta(t)<1$, $\frac{\partial^{\alpha(x)}u(x,t)}{\partial x^{\alpha(x)}}$ and $\frac{\partial^{\beta(t)}v(x,t)}{\partial t^{\beta(t)}}$ are derivative of Caputo definition. $u(x,t)$, $v(x,t)$, $g_1(u,v)$, $g_2(u,v)$, $f_1(x,t)$, $f_2(x,t)$ are square integrable functions on $[0,X]\times[0,T]$. And $g_1(u,v)$, $g_2(u,v)$, $f_1(x,t)$, $f_2(x,t)$ are known functions, $u(x,t)$, $v(x,t)$ are pending functions.

The structure of this paper is as follows: In Section 2, we introduce some definitions of shifted Chebyshev polynomials and variable-order fractional differential operators. In Section 3, we present the algorithm and differential operator matrix. Error analysis of approximate solutions is mainly investigated in Section 4. Three illustrative numerical examples are shown to demonstrate the accuracy and efficiency of the proposed method in Section 5. Finally, conclusions are given in Section 6.

2. PRELIMINARIES

2.1. Fractional Calculus and Shifted Chebyshev Polynomials

Definition 1 The Riemann - Liouville fractional derivative of the variable order $\alpha(t)$ is defined as follows:

$$ {}^{RL}D_t^{\alpha(t)}f(t)=\frac{1}{\Gamma(n-\alpha(t))}\frac{d^n}{dt^n}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha(t)-n+1}}d\tau, n-1<\alpha(t)<n. \quad (3)$$

Definition 2 The Caputo fractional derivative of the variable order $\alpha(t)$ is defined as follows:

$$ {}^{C}D_t^{\alpha(t)}f(t)=\frac{1}{\Gamma(n-\alpha(t))}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha(t)-n+1}}d\tau, n-1<\alpha(t)<n. \quad (4)$$

Definition 3 The Chebyshev polynomials on the interval $[-1,1]$ are defined as follows:

$$\begin{gathered} G_{i+1}(x)=2xG_i(x)-G_{i-1}, \quad i=1,2,3,\cdots \\ G_0(x)=1, \quad G_1(x)=x. \end{gathered} \quad (5)$$

The shifted Chebyshev polynomials are defined on $[0,R]$ by taking transformation $x=\frac{2t}{R}-1$ in the ones obtained on $[-1,1]$. Therefore, the shifted Chebyshev polynomials on the interval $[0,R]$ are given by:

$$\begin{gathered} T_{i+1}(t)=2\left(\frac{2t}{R}-1\right)T_i(t)-T_{i-1}(t), i=1,2,3\cdots \\ T_0(t)=1, T_1(t)=\frac{2t}{R}-1. \end{gathered} \quad (6)$$

The explicit form of the shifted Chebyshev polynomials $T_n(t)$ of degree n is given as follows:

$$T_n(t)=\begin{cases} 1, & n=0, \\ \sum_{k=0}^{n}(-1)^{n-k}4^k\frac{n(n+k-1)}{(n-k)!(2k)!}t^k, & n\geq 1. \end{cases} \quad (7)$$

The shifted Chebyshev polynomials are orthogonal on $[0,R]$ with respect to the weight function $\omega(t)=\frac{1}{\sqrt{Rt-t^2}}$. The orthogonality conditions are presented by the following relations:

$$\int_0^R T_n(t)T_m(t)\omega(t)dt=\begin{cases} 0, & n\neq m, \\ \pi, & n=m=0, \\ \frac{\pi}{2}, & n=m\neq 0. \end{cases} \quad (8)$$

Now, we define:

$$\Phi(t)=[T_0(t),T_1(t),\cdots,T_n(t)]^T, \quad (9)$$

then we get:

$$\Phi(t)=A_RT(t), \quad (10)$$

where

$$T(t)=[1,t,t^2,\ldots,t^n]^T, \quad (11)$$

$$A_R=\begin{bmatrix} P_{0,0} & 0 & \cdots & 0 \\ P_{1,0} & P_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{n,0} & P_{n,1} & \cdots & P_{n,n} \end{bmatrix}, \quad (12)$$

where

$$\begin{cases} P_{0,0}=1, \\ P_{i,j}=2(\frac{2}{R}P_{i-1,j-1}-P_{i-1,j})-P_{i-2,j}, \\ P_{i,j}=0, \text{for } i<j \text{ or } i<0 \text{ or } j<0. \end{cases}$$

Obviously, A_R is full rank and reversible.

2.2. Function Approximation

The approximation function based on the families of SCPs of x and t was applied to replace the Unknown function. The collocation method was used to discretize the variables x and t to transform the set of variable-order fractional partial differential equations into a set of algebraic equations. The numerical algorithm is summarized in Figure **1**.

For arbitrary functions $u(x,t), v(x,t)\in L^2([0,X]\times[0,T])$, they can be developed in terms of the shifted Chebyshev polynomials. By taking the $(n+1)^2$ first shifted Chebyshev polynomials, we get:

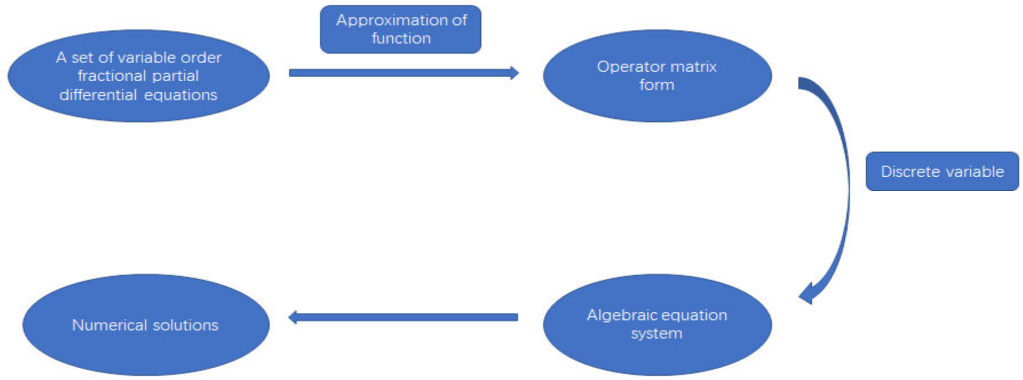

Figure 1: A schematic illustration of the proposed numerical algorithm.

$$u(x,t) \approx u_n(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} u_{i,j}T_i(x)T_j(t) = \Phi^T(x)C\Phi(t), \quad (13)$$

$$v(x,t) \approx v_n(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} v_{i,j}T_i(x)T_j(t) = \Phi^T(x)K\Phi(t), \quad (14)$$

where $u_{i,j}, v_{i,j} (i = 0,1,\cdots,n; j = 0,1,\cdots,n)$ are called two-dimensional functions, C and K are the shifted Chebyshev polynomials approximation coefficients of $u(x,t)$ and $v(x,t)$, respectively.

3. SHIFTED CHEBYSHEV POLYNOMIALS DIFFERENTIAL OPERATOR MATRIX

3.1. First Order Differential Operator Matrix of Shifted Chebyshev Polynomials

Definition 4 If there are matrices D and P, satisfying $\Phi'(t) = D\Phi(t)$ and $\Phi'(x) = P\Phi(x)$, then D and P are called the first-order differential operator matrices of shifted Chebyshev polynomials.

By calculating the first-order derivative of $\Phi(t)$, we get:

$$\Phi'(t) = \left(A_T T(t)\right)' = A_T T'(t)$$
$$= A_T \begin{bmatrix} 1' \\ t' \\ \vdots \\ (t^n)' \end{bmatrix} = A_T \begin{bmatrix} 0 \\ 1 \\ \vdots \\ nt^{n-1} \end{bmatrix} = A_T V_{(n+1)\times n} T^*(t), \quad (15)$$

where A_T is obtained by replacing R by T in Eq. (12), and

$$V_{(n+1)\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & n \end{bmatrix}, T^*(t) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{n-2} \\ t^{n-1} \end{bmatrix}. \quad (16)$$

By using Eq. (10), we obtain:

$$T^*(t) = B_T^*\Phi(t), \quad (17)$$

where $B_T^* = \left[A_{T,[1]}^{-1}, A_{T,[2]}^{-1}, \cdots, A_{T,[n]}^{-1}\right]^T$.

According to Eq. (15) and (17), we get:

$$\Phi'(t) = A_T V_{(n+1)\times n} B_T^* \Phi(t) = D\Phi(t), \quad (18)$$

where $D = A_T V_{(n+1)\times n} B_T^*$ is a first-order differential operator matrix of the shifted Chebyshev polynomials.

The similar result can be obtained:

$$\Phi'(x) = P\Phi(x), \quad (19)$$

where $P = A_X V_{(n+1)\times n} B_X^*$.

Now, using Eqs. (13), (14), (18) and (19), we get:

$$\frac{\partial u(x,t)}{\partial x} \approx \frac{\partial\left(\Phi^{\mathrm{T}}(x)C\Phi(t)\right)}{\partial x}$$
$$= \Phi^{\mathrm{T}}(x)P^{\mathrm{T}}C\Phi(t)$$
$$= \Phi^{\mathrm{T}}(x)\left(A_X V_{(n+1)\times n} B_X^*\right)^{\mathrm{T}} C\Phi(t) \quad (20)$$

$$\frac{\partial v(x,t)}{\partial t} \approx \frac{\partial\left(\Phi^{\mathrm{T}}(x)K\Phi(t)\right)}{\partial t}$$
$$= \Phi^{\mathrm{T}}(x)KD\Phi(t)$$
$$= \Phi^{\mathrm{T}}(x)KA_T V_{(n+1)\times n} B_T^*\Phi(t) \quad (21)$$

Furthermore, the higher-order differential operator matrices derived from SCPs by mathematical induction have the following form:

$$\frac{d^n}{dx^n}\Phi(x) = (P)^n\Phi(x). \quad (22)$$

3.2. Variable order differential operator matrix of the shifted Chebyshev polynomials

Definition 5 If there are matrices $P_t^{\beta(t)}$ and $P_x^{\alpha(x)}$, satisfying $\frac{\partial^{\beta(t)}\Phi(t)}{\partial t^{\beta(t)}} = P_t^{\beta(t)}\Phi(t)$ and

$\frac{\partial^{\alpha(x)}\Phi^{\mathrm{T}}(x)}{\partial x^{\alpha(x)}} = \Phi^{\mathrm{T}}(x)(P_x^{\alpha(x)})^{\mathrm{T}}$, then $P_t^{\beta(t)}$ and $P_x^{\alpha(x)}$ are called variable order differential operator matrices of shifted Chebyshev polynomials.

By taking the variable-order fractional derivative of $v(x,t)$, we obtain:

$$
\begin{aligned}
&\frac{\partial^{\beta(t)}v(x,t)}{\partial t^{\beta(t)}}\\
&\approx \frac{\partial^{\beta(t)}\left(\Phi^{\mathrm{T}}(x)K\Phi(t)\right)}{\partial t^{\beta(t)}}3ex\\
&= \Phi^{\mathrm{T}}(x)K\frac{\partial^{\beta(t)}\Phi(t)}{\partial t^{\beta(t)}}3ex\\
&= \Phi^{T}(x)K\left[0,\cdots,\frac{\Gamma(\beta(t)+1)}{\Gamma(\beta(t)+1-\beta(t))}t^{\beta(t)-\beta(t)},\cdots,\frac{\Gamma(i+1)}{\Gamma(i+1-\beta(t))}t^{i-\beta(t)},\cdots\right]^{\mathrm{T}}\\
&= \Phi^{\mathrm{T}}(x)KA_T V^{*}_{(n+1)\times(n+1)}A_T^{-1}\Phi(t)\\
&= \Phi^{\mathrm{T}}(x)KP_t^{\beta(t)}\Phi(t)3ex,
\end{aligned}
\tag{23}
$$

where $i = \beta(t), \beta(t)+1, \cdots, n$, $P_t^{\beta}(t) = A_T V^{*}_{(n+1)\times(n+1)}A_T^{-1}$, and

$$
V^{*}_{(n+1)\times(n+1)} =
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \frac{\Gamma(\beta(t)+1)t^{-\beta(t)}}{\Gamma(\beta(t)+1-\beta(t))} & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & \cdots & 0 & \cdots & \frac{\Gamma(i+1)t^{-\beta(t)}}{\Gamma(i+1-\beta(t))} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{\Gamma(n+1)t^{-\beta(t)}}{\Gamma(n+1-\beta(t))}
\end{bmatrix}.
\tag{24}
$$

The fractional-order differential operator matrix $P_t^{\beta(t)}$ is:

$$
P_t^{\beta(t)} =
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
S_\beta(\beta(t),0) & \cdots & S_\beta(\beta(t),\beta(t)) & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
S_\beta(i,0) & \cdots & S_\beta(i,\beta(t)) & \cdots & S_\beta(i,i) & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
S_\beta(n,0) & \cdots & S_\beta(n,\beta(t)) & \cdots & S_\beta(n,i) & \cdots & S_\beta(n,n)
\end{bmatrix}.
\tag{25}
$$

The similar result can be obtained:

$$
\begin{aligned}
\frac{\partial^{\alpha(x)}u(x,t)}{\partial x^{\alpha(x)}} &\approx \frac{\partial^{\alpha(x)}\left(\Phi^{\mathrm{T}}(x)C\Phi(t)\right)}{\partial x^{\alpha(x)}}3ex\\
&= \left(\frac{\partial^{\alpha(x)}\Phi(x)}{\partial x^{\alpha(x)}}\right)^{\mathrm{T}} C\Phi(t)3ex\\
&= (A_X V^{**}_{(n+1)\times(n+1)}A_X^{-1}\Phi(\mathrm{x}))^{\mathrm{T}}C\Phi(t)\\
&= \Phi^{\mathrm{T}}(x)(P_x^{\alpha(x)})^{\mathrm{T}}C\Phi(t)3ex,
\end{aligned}
\tag{26}
$$

where $i = \alpha(x), \alpha(x)+1, \cdots, n$, $P_x^{\alpha(x)} = A_X V^{**}_{(n+1)\times(n+1)}A_X^{-1}$, and

$$
V^{**}_{(n+1)\times(n+1)} =
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & \frac{\Gamma(\alpha(x)+1)x^{-\alpha(x)}}{\Gamma(\alpha(x)+1-\alpha(x))} & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & \cdots & 0 & \cdots & \frac{\Gamma(i+1)t^{-\alpha(x)}}{\Gamma(i+1-\alpha(x))} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & \cdots & 0 & \cdots & \frac{\Gamma(n+1)x^{-\alpha(x)}}{\Gamma(n+1-\alpha(x))}
\end{bmatrix}
\tag{27}
$$

The fractional-order differential operator matrix $P_x^{\alpha(x)}$ is:

$$
P_x^{\alpha(x)} =
\begin{bmatrix}
0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
S_\alpha(\alpha(x),0) & \cdots & S_\alpha(\alpha(x),\alpha(x)) & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
S_\alpha(i,0) & \cdots & S_\alpha(i,\alpha(x)) & \cdots & S_\alpha(i,i) & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
S_\alpha(n,0) & \cdots & S_\alpha(n,\alpha(x)) & \cdots & S_\alpha(n,i) & \cdots & S_\alpha(n,n)
\end{bmatrix}
\tag{28}
$$

$$
\begin{cases}
\Phi^{\mathrm{T}}(x)\left(A_X{}^{-1}\right)^{\mathrm{T}}V^{**}_{(n+1)\times(n+1)}A_X{}^{\mathrm{T}}C\Phi(t)+\Phi^{\mathrm{T}}(x)\\
\quad (A_X V_{(\mathrm{n}+1)\times n}B_X^{*})^{\mathrm{T}}C\Phi(t)+g_1(u,v)=f_1(x,t),\\
\Phi^{\mathrm{T}}(x)V^{*}_{(n+1)\times(n+1)}A_T F A_T{}^{-1}\Phi(t)+\Phi^{\mathrm{T}}(x)\\
\quad V_{(\mathrm{n}+1)\times n}B_T^{*}\Phi(t)+g_2(u,v)=f_2(x,t),
\end{cases}
\tag{29}
$$

Eqs. (20), (21), (27) and (28) are substituted into Eq. (1):

then the initial conditions and boundary conditions are converted into:

$$
\begin{aligned}
&u(0,t)\approx\Phi^{T}(0)C\Phi(t), u(x,0)\approx\Phi^{T}(x)C\Phi(0),\\
&v(0,t)\approx\Phi^{T}(0)K\Phi(t), v(x,0)\approx\Phi^{T}(x)K\Phi(0).
\end{aligned}
\tag{30}
$$

Divide the variable order (x,t) by the points $(x_i,t_j)=\left(X\left(\frac{2i-1}{2(n+2)}\right),\,T\left(\frac{2j-1}{2(n+2)}\right)\right)$ $(i=1,2,\cdots,N,$ $j=1,2,\cdots,N)$ in Eqs. (29) and (30). We can get the coefficients matrices C and K by using Matlab, so that the approximate solutions of $u(x,t)$ and $v(x,t)$ can be found.

4. ERROR CORRECTION

In this section, the error analysis of the approximate solutions for the considered system of variable-order fractional partial differential equations is carried out based on the shifted Chebyshev polynomials. First, with the aid of the residual functions and the error functions, the estimation errors for approximate solutions are obtained. Then, the approximate error functions are considered to correct the approximate solutions. Finally, we obtain the correction solutions and improve the accuracy of approximate solutions.

We consider the following sets of residual functions $R_n^u(x,t)$ and $R_n^v(x,t)$:

$$R_n^u(x,t)=L[u_n(x,t)]+f_1(x,t), \tag{31}$$

$$R_n^v(x,t)=L[v_n(x,t)]+f_2(x,t), \tag{32}$$

where $u_n(x,t)$, $v_n(x,t)$ are the approximate solutions, and $L[u_n(x,t)]$ and $L[v_n(x,t)]$ are defined by:

$$L[u_n(x,t)]=\frac{\partial^{\alpha(x)}u_n(x,t)}{\partial x^{\alpha(x)}}+\frac{\partial u_n(x,t)}{\partial x}+g_1(u_n,v_n), \tag{33}$$

and

$$L[v_n(x,t)]=\frac{\partial^{\beta(t)}v_n(x,t)}{\partial t^{\beta(t)}}+\frac{\partial v_n(x,t)}{\partial t}+g_2(u_n,v_n). \tag{34}$$

Then, $L[u_n(x,t)]$ and $L[v_n(x,t)]$) can be expressed as:

$$L[u_n(x,t)]=R_n^u(x,t)-f_1(x,t), \tag{35}$$

$$L[v_n(x,t)]=R_n^v(x,t)-f_2(x,t). \tag{36}$$

Define the following error functions:

$$e_n^u=u(x,t)-u_n(x,t), e_n^v=v(x,t)-v_n(x,t), \tag{37}$$

where $u(x,t)$ and $v(x,t)$ are the exact solutions of the set of variable-order fractional partial differential equations. Thus, the differential equations of the error functions are obtained as follows:

$$\begin{aligned} L[e_n^u(x,t)] &= L[u(x,t)]-L[u_n(x,t)] \\ &= 0-f_1(x,t)-R_n^u(x,t)+f_1(x,t) \\ &= -R_n^u(x,t), \end{aligned} \tag{38}$$

and

$$L[e_n^v(x,t)]=-R_n^v(x,t). \tag{39}$$

Then, we obtain:

$$L[e_n^u(x,t)]=\frac{\partial^{\alpha(x)}e_n^u(x,t)}{\partial x^{\alpha(x)}}+\frac{\partial e_n^u(x,t)}{\partial x}+g_1(e_n^u,e_n^v)=-R_n^u(x,t), \tag{40}$$

$$L[e_n^v(x,t)]=\frac{\partial^{\beta(t)}e_n^v(x,t)}{\partial t^{\beta(t)}}+\frac{\partial e_n^v(x,t)}{\partial t}+g_2(e_n^u,e_n^v)=-R_n^v(x,t), \tag{41}$$

where $e_n^u(x,t)$ and $e_n^v(x,t)$ can be approximated by $e_u^*(x,t)$ and $e_v^*(x,t)$ using the algorithm proposed in the previous section.

Therefore, the correct solutions can be obtained:

$$u'(x,t)=u_n(x,t)+e_u^*(x,t), v'(x,t)=v_n(x,t)+e_v^*(x,t). \tag{42}$$

Furthermore, we can get:

$$\begin{aligned} E_u(x,t) &= e_n^u(x,t)-e_u^*(x,t)=u(x,t)-u'(x,t) \\ &= u(x,t)-u_n(x,t)-e_v^*(x,t), \end{aligned} \tag{43}$$

$$\begin{aligned} E_v(x,t) &= e_n^v(x,t)-e_v^*(x,t)=v(x,t)-v'(x,t) \\ &= v(x,t)-v_n(x,t)-e_v^*(x,t), \end{aligned} \tag{44}$$

where $E_u(x,t)$ and $E_v(x,t)$ are called correct error functions.

5. NUMERICAL SIMULATION

In this section, we apply the proposed method to solve three systems of variable-order fractional partial differential equations.

Example 1 Let us consider the following system of partial differential equations with variable-order fractional derivatives:

$$\begin{cases} \dfrac{\partial^{\frac{1+\cos x}{3}}u(x,t)}{\partial x^{\frac{1+\cos x}{3}}}+\dfrac{\partial u(x,t)}{\partial x}+u(x,t)-2v(x,t)=f_1(x,t), \\ \dfrac{\partial^{\frac{t+2}{5}}v(x,t)}{\partial t^{\frac{t+2}{5}}}+\dfrac{\partial v(x,t)}{\partial t}+2u(x,t)-v(x,t)=f_2(x,t), \end{cases} \tag{45}$$

with the following initial and boundary conditions on $[0,2]\times[0,3]$:

$$u(x,0)=10x(1-x), u(0,t)=0, v(x,0)=0, v(0,t)=10t(t-1), \tag{46}$$

where

$$\begin{cases} f_1(x,t) = 10(1+t)^2 \left(\dfrac{x^{\frac{2-\cos x}{3}}}{\Gamma\left(\dfrac{5-\cos x}{3}\right)} - \dfrac{2x^{\frac{5-\cos x}{3}}}{\Gamma\left(\dfrac{8-\cos x}{3}\right)} - x^2 - x + 1 \right) \\ \qquad - 20(1+x)^2(1-t)\,t, \\ f_2(x,t) = 10(1+x)^2 \left(\dfrac{t^{\frac{3-t}{5}}}{\Gamma\left(\dfrac{8-t}{5}\right)} - \dfrac{2x^{\frac{8-t}{5}}}{\Gamma\left(\dfrac{13-t}{5}\right)} + t^2 - 3t + 1 \right) \\ \qquad - 20(1+t)^2(1-x)\,x. \end{cases} \tag{47}$$

The exact solutions of the equations are given by:

$$\begin{cases} u(x,t) = 10x(1-x)(1+t)^2, \\ v(x,t) = 10t(1-t)(1+x)^2. \end{cases} \tag{48}$$

Taking t = 1.5, the approximate and exact solutions are given in Figure **2**. Figure **3** shows the absolute errors between approximate and exact solutions on $[0,2]\times[0,3]$. Absolute errors and correct errors are presented at different times in Figure **4** to **5**.

From Figure **2** and **3**, we can conclude that the approximate solutions of the system converge to the exact solutions very well. Figure **4** to **5** also show that the correct errors are smaller.

Example 2 Let us consider the following set of variable order fractional partial differential equations:

$$\begin{cases} \dfrac{\partial^{\frac{x-1}{3}} u(x,t)}{\partial x^{\frac{x-1}{3}}} + \dfrac{\partial u(x,t)}{\partial x} - u(x,t) = f_1(x,t), \\ \dfrac{\partial^{\frac{\cos t+1}{5}} v(x,t)}{\partial t^{\frac{\cos t+1}{5}}} + \dfrac{\partial v(x,t)}{\partial t} - v(x,t) = f_2(x,t), \end{cases} \tag{49}$$

subject to the initial and boundary conditions on $[0,3]\times[0,4]$:

$$u(x,0) = 5x(x^2-1), u(0,t) = 0, v(x,0) = 0, v(0,t) = 5t(t^2-1), \tag{50}$$

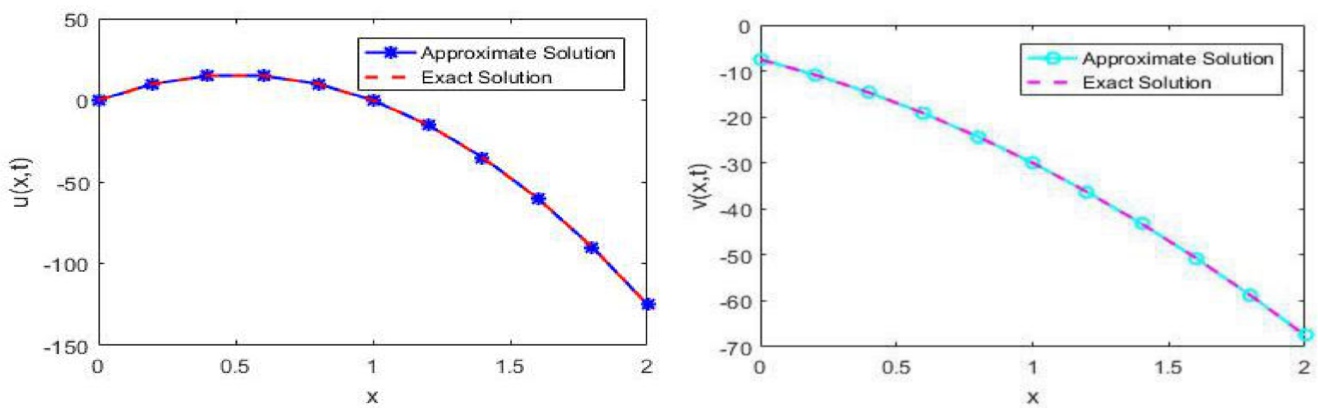

Figure 2: Approximate and exact solutions for Example 1, when $n = 2$.

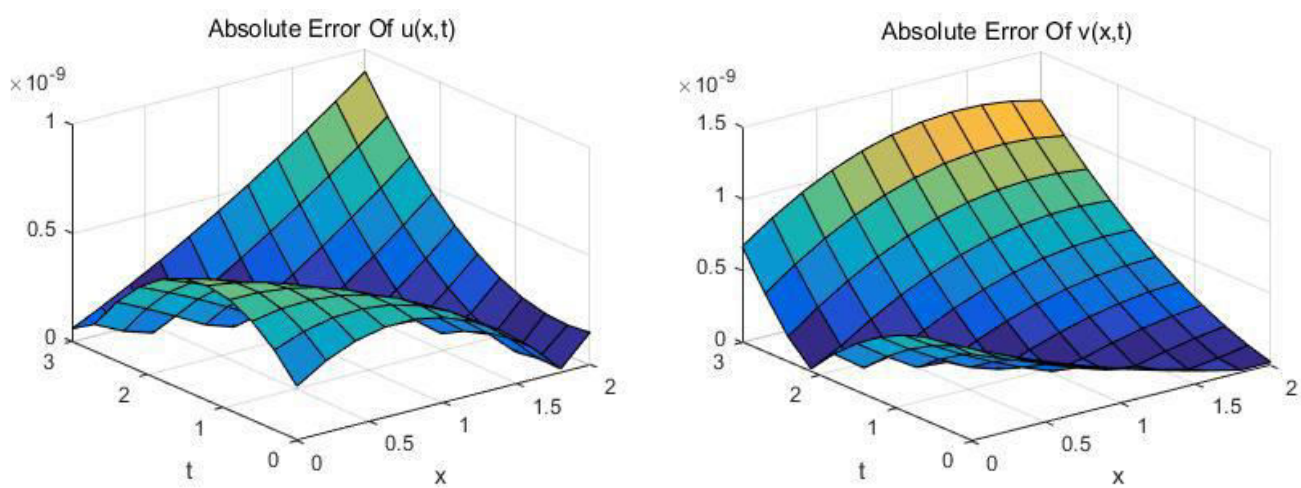

Figure 3: Absolute errors between approximate and exact solutions for Example 1, when $n = 2$.

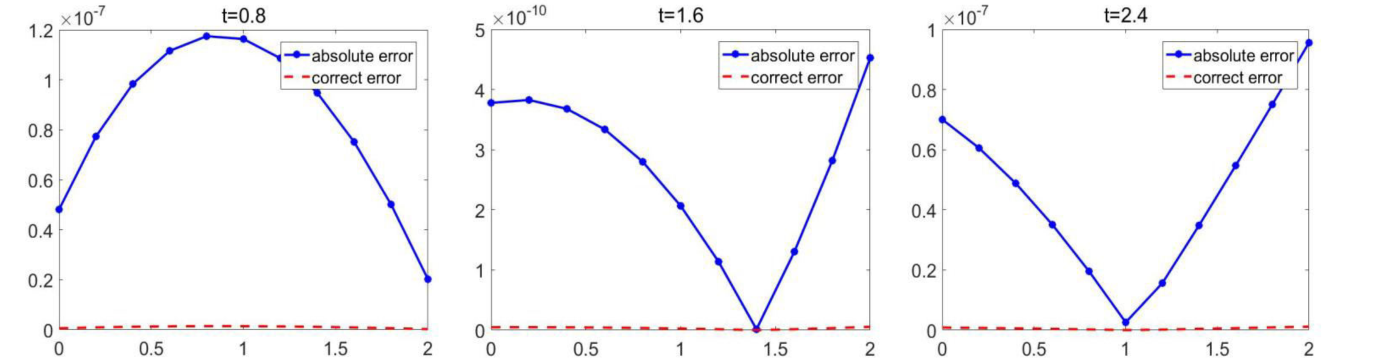

Figure 4: Absolute errors and correct errors of $u(x,t)$ for Example 1, when $n=2$.

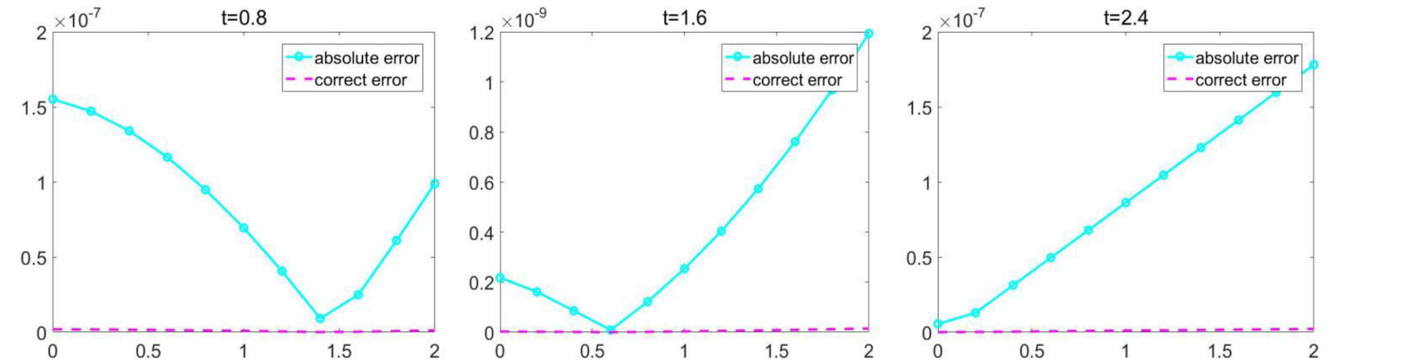

Figure 5: Absolute errors and correct errors of $v(x,t)$ for Example 1, when $n=2$.

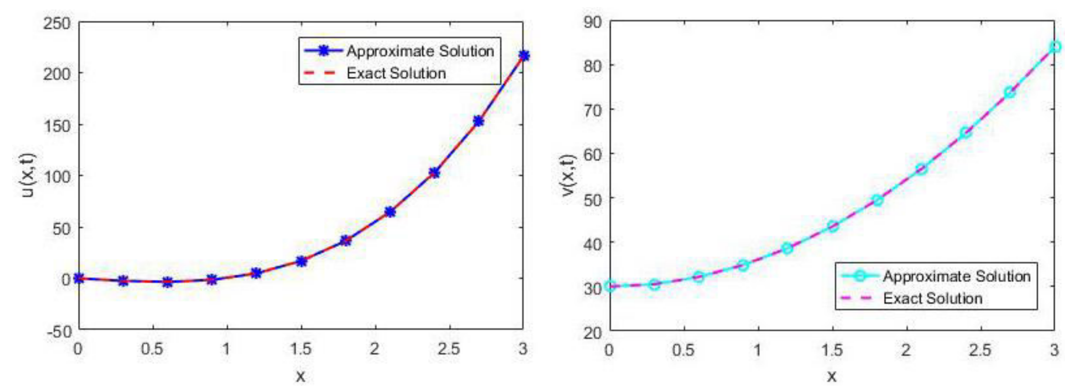

Figure 6: Approximate and exact solutions for Example 2, when $n=3$.

where

$$\begin{cases} f_1(x,t) = \left(\dfrac{6x^{\frac{10-x}{3}}}{\Gamma\left(\dfrac{13-x}{3}\right)} - \dfrac{x^{\frac{4-x}{3}}}{\Gamma\left(\dfrac{7-x}{3}\right)} + 3x^2 - 1 \right) \\ \qquad \times \left(5+t^2+t^3\right) - \left(5+t^2+t^3\right)\left(x^3-x\right), \\ f_2(x,t) = \left(\dfrac{6t^{\frac{11-\cos t}{4}}}{\Gamma\left(\dfrac{15-\cos t}{4}\right)} - \dfrac{t^{\frac{3-\cos t}{4}}}{\Gamma\left(\dfrac{7-\cos t}{4}\right)} + 3t^2 + 1 \right) \\ \qquad \times \left(5+x^2+x^3\right) - \left(5+x^2+x^3\right)\left(t^3-t\right). \end{cases} \tag{51}$$

The exact solutions for the set of equations are:

$$\begin{cases} u(x,t) = x(x^2-1)(5+t^2+t^3), \\ v(x,t) = t(t^2-1)(5+x^2+x^3). \end{cases} \tag{52}$$

Figure **6** presents the approximate and exact solutions at $t=2$. The absolute errors and correct errors for some nodes on $[0,3]\times[0,4]$ are shown in Figure **7** to **9**. The absolute errors and correct errors with $n=3$, $n=4$, $n=5$ at $t=0.8,1.6,2.4,3.6$ are computed in Table **1**.

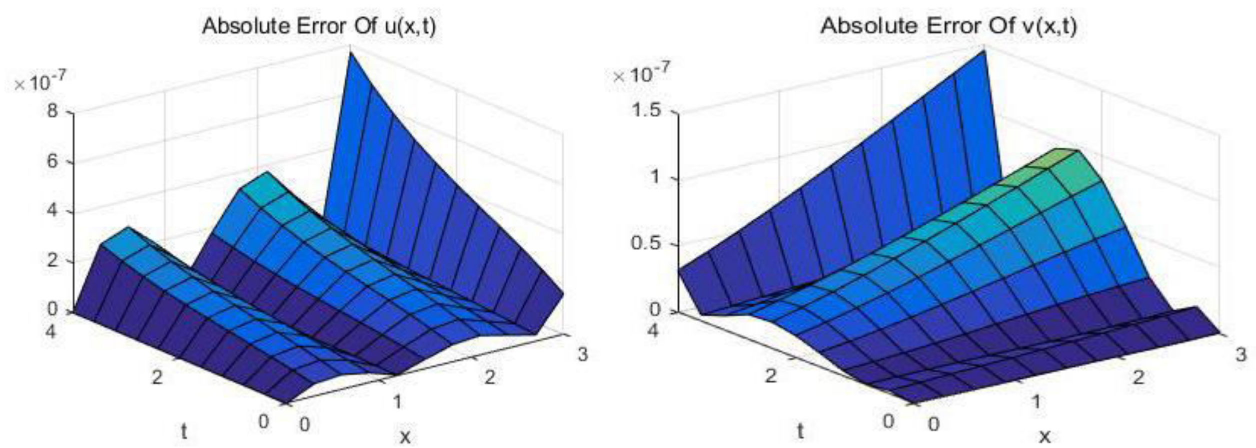

Figure 7: Absolute errors between approximate and exact solutions for Example 2, when $n=3$.

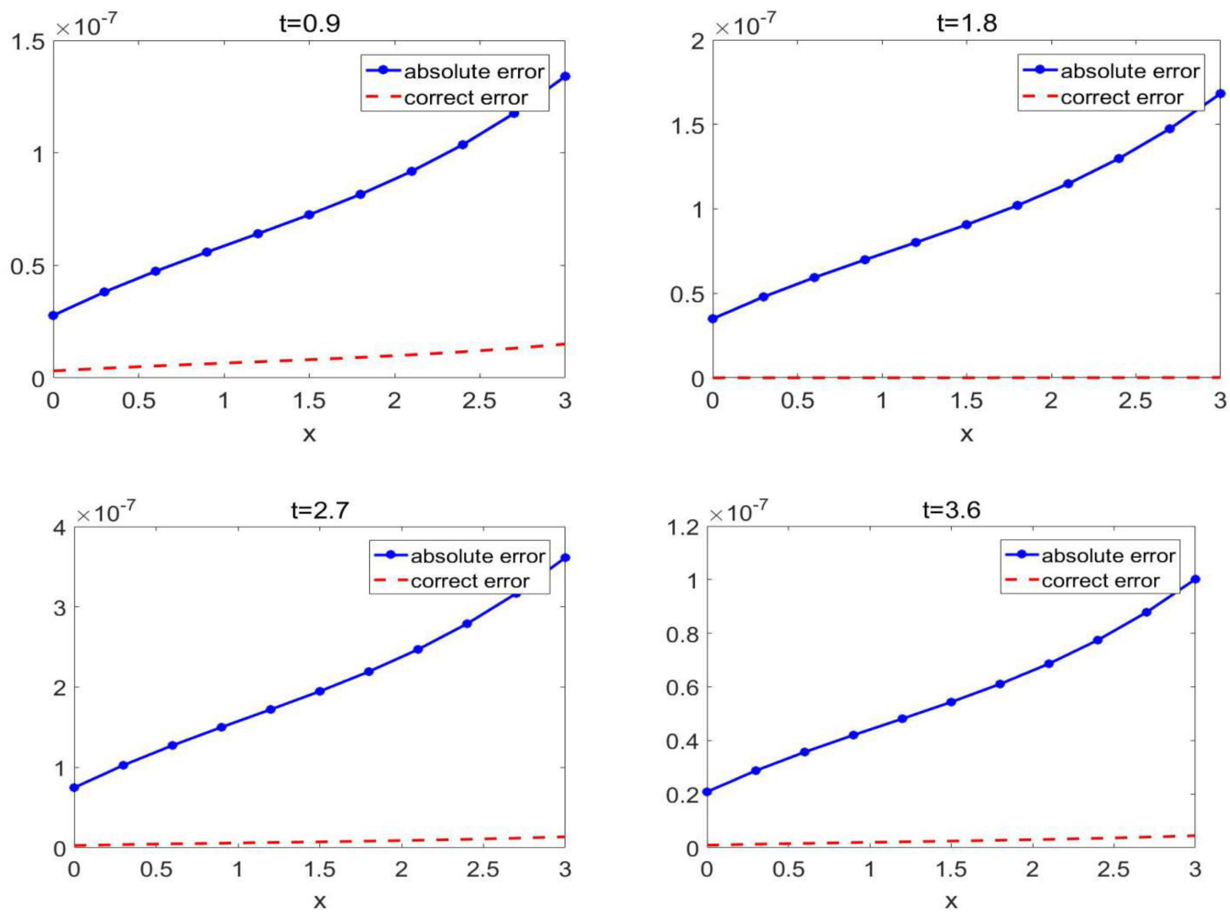

Figure 8: Absolute errors and correct errors of $u(x,t)$ for Example 2, when $n=3.$.

Figure **6** shows that the approximate solutions approach to the exact solutions. It is obvious from Figure **7** to **9** and Table **1** that the absolute errors and correct errors can be reduced by increasing the values of n.

Example 3 Consider the following equations:

$$\begin{cases} \frac{\partial^{\alpha(x)}u(x,t)}{\partial x^{\alpha(x)}} + \frac{\partial u(x,t)}{\partial x} + u(x,t) + v(x,t) = f_1(x,t), \\ \frac{\partial^{\beta(t)}v(x,t)}{\partial t^{\beta(t)}} + \frac{\partial v(x,t)}{\partial t} + u(x,t) - v(x,t) = f_2(x,t), \end{cases} \tag{53}$$

with the initial conditions and boundary conditions $u(x,0) = u(0,t) = 0$, $v(x,0) = v(0,t) = 0$ on $[0,2] \times [0,3]$, where

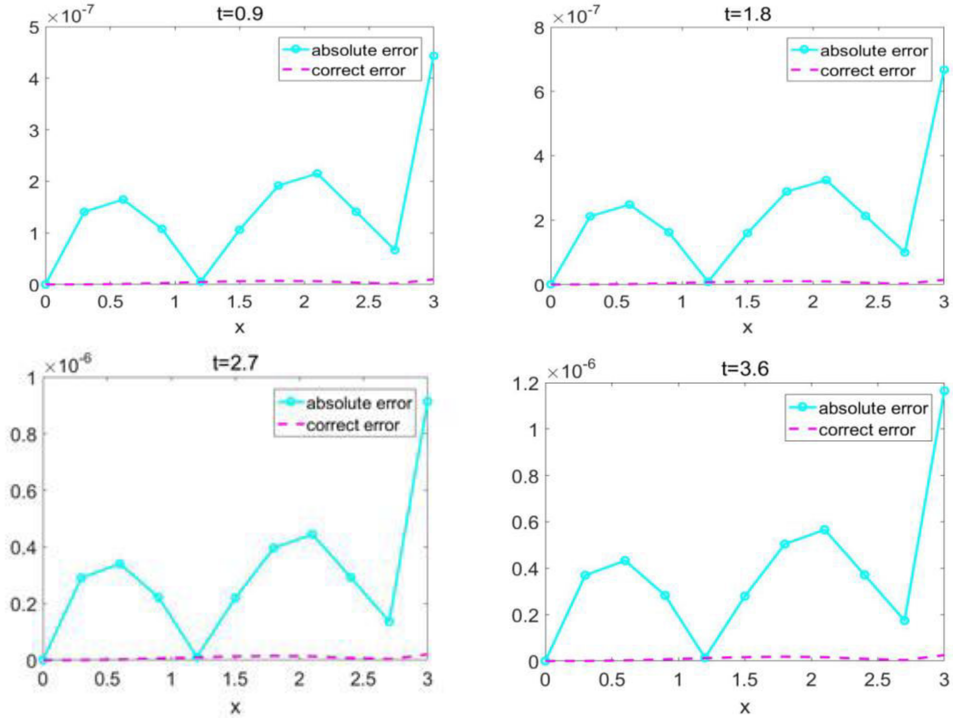

Figure 9: Absolute errors and correct errors of $v(x,t)$ for Example 2, when $n=3$.

Table 1: The Comparisons between Absolute Errors $e_u(x,t)$, $e_v(x,t)$ and Correct Errors $e_u^*(x,t)$, $e_v^*(x,t)$ for Example 2

t	*x*	$n=3$				$n=4$				$n=5$			
		$e_u(x,t)$	$e_u^*(x,t)$	$e_v(x,t)$	$e_v^*(x,t)$	$e_u(x,t)$	$e_u^*(x,t)$	$e_v(x,t)$	$e_v^*(x,t)$	$e_u(x,t)$	$e_u^*(x,t)$	$e_v(x,t)$	$e_v^*(x,t)$
0.8	0.6	1.02e-07	4.07e-10	9.35e-09	4.11e-09	1.82e-08	6.25e-10	3.45e-09	2.37e-10	2.27e-11	4.11e-12	1.72e-09	1.16e-09
	1.8	1.18e-07	3.67e-09	1.99e-09	8.09e-09	1.99e-08	2.19e-09	6.79e-09	4.64e-10	9.68e-11	1.82e-11	3.48e-09	2.29e-09
	2.7	4.02e-08	8.99e-10	4.73e-08	1.12e-08	1.51e-09	1.31e-12	9.44e-09	6.49e-10	2.22e-10	5.67e-12	4.90e-09	3.17e-09
1.6	0.6	1.39e-07	5.35e-10	1.83e-08	1.62e-10	2.37e-08	8.60e-10	5.04e-10	2.98e-11	5.74e-11	5.66e-12	2.18e-10	9.38e-10
	1.8	1.58e-07	4.95e-09	3.73e-08	3.72e-10	2.73e-08	2.96e-09	8.39e-10	5.94e-11	5.37e-11	2.51e-11	4.26e-09	1.85e-09
	2.7	5.48e-08	1.23e-09	5.17e-08	5.59e-10	1.44e-09	2.06e-11	1.04e-09	8.57e-11	3.30e-10	7.79e-12	5.83e-09	2.56e-09
2.4	0.6	1.77e-07	6.61e-10	3.32e-08	5.33e-09	2.96e-08	1.10e-09	2.55e-09	1.22e-10	9.37e-11	7.20e-12	4.78e-09	2.05e-09
	1.8	2.01e-07	6.31e-09	6.53e-08	1.05e-08	3.52e-08	3.77e-09	3.41e-09	2.45e-10	8.92e-12	3.19e-11	9.92e-09	4.05e-09
	2.7	6.97e-08	1.60e-09	9.18e-08	1.46e-08	9.11e-10	8.19e-11	7.69e-09	3.44e-10	4.45e-10	9.97e-12	1.41e-09	5.60e-09
3.6	0.6	2.54e-07	9.30e-10	7.43e-09	1.74e-09	4.17e-08	1.56e-09	1.64e-10	2.92e-12	1.72e-10	1.03e-11	4.57e-10	4.48e-10
	1.8	2.89e-07	9.09e-09	1.25e-08	3.51e-09	5.16e-08	5.41e-09	1.88e-09	1.19e-11	4.61e-11	4.61e-11	8.79e-09	8.87e-10
	2.7	1.01e-07	2.26e-09	1.58e-08	4.88e-09	5.52e-10	1.68e-10	3.88e-11	4.54e-12	1.48e-11	1.48e-11	1.21e-09	1.20e-09

$$\begin{cases} f_1(x,t) = t\left(\dfrac{2x^{\frac{6-x}{3}}}{\Gamma\left(\dfrac{9-x}{3}\right)} - \dfrac{x^{\frac{3-x}{3}}}{\Gamma\left(\dfrac{6-x}{3}\right)} - 1\right) + xt(x+t), \\ f_2(x,t) = x\left(\dfrac{2t^{\frac{8-t}{4}}}{\Gamma\left(\dfrac{12-t}{4}\right)} - \dfrac{t^{\frac{4-t}{4}}}{\Gamma\left(\dfrac{8-t}{4}\right)} - 1\right) + xt(x-t+2) \end{cases} \tag{54}$$

The exact solutions of the equations for $\alpha(x) = \frac{x}{3}$, $\beta(t) = \frac{t}{4}$ are:

$$\begin{cases} u(x,t) = xt(x-1), \\ v(x,t) = xt(t-1). \end{cases} \tag{55}$$

Figure **10** presents the approximate and exact solutions at $t = 1.5$. Figure **11** to **13** show the absolute

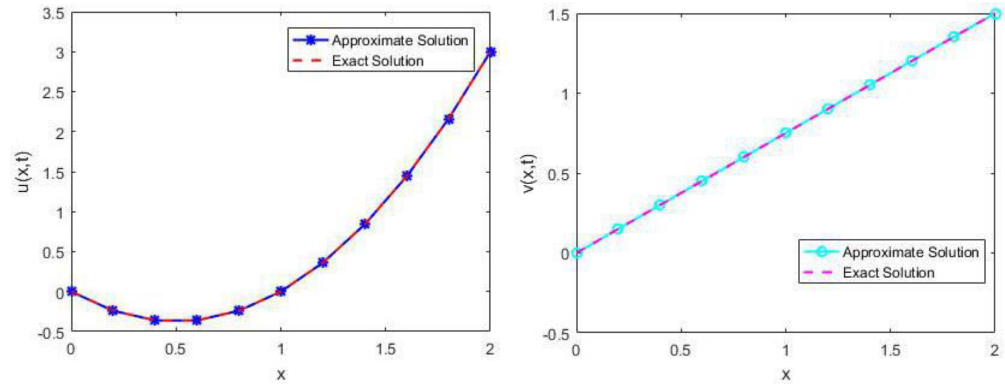

Figure 10: Approximate and exact solutions for Example 3, when $n = 2$.

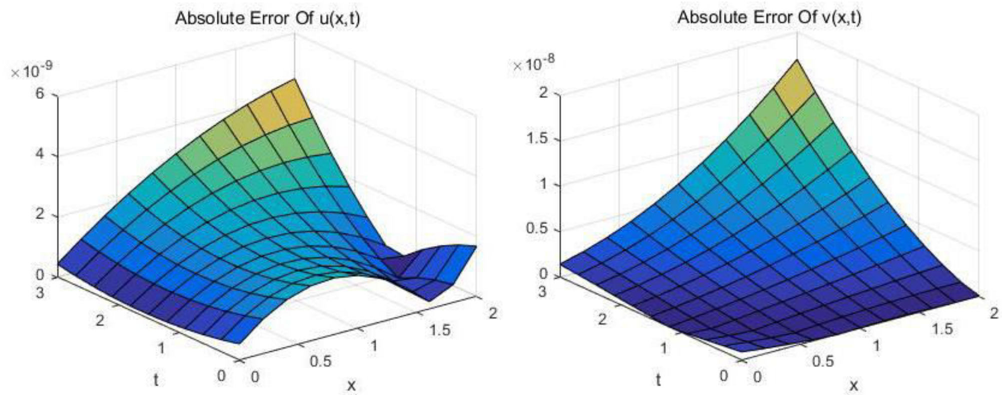

Figure 11: Absolute errors between approximate and exact solutions for Example 3, when $n = 2$.

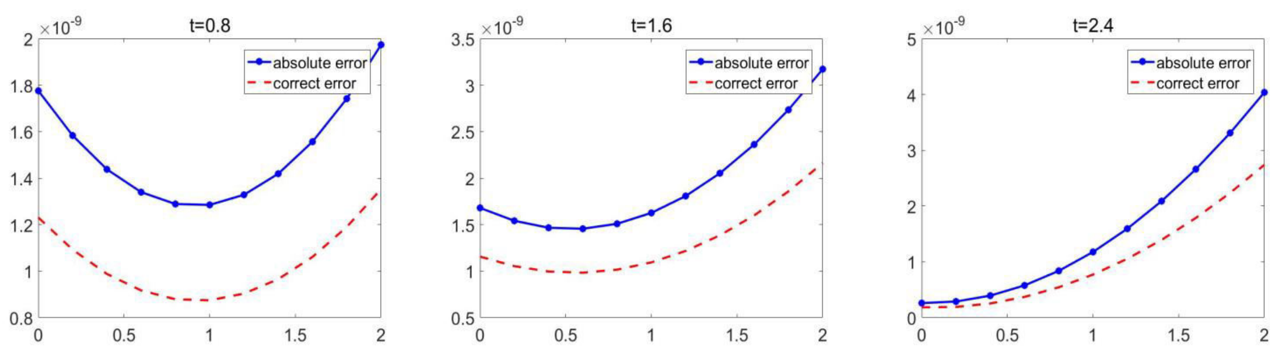

Figure 12: Absolute errors and correct errors of $u(x,t)$ for Example 3, when $n = 2$.

errors and correct errors for some values of $t = 0.8, 1.6, 2.4$ with $n = 2$. The absolute errors and correct errors for $n = 3$ are shown in Figure **14** and **15**.

From Figure **10** to **15**, we can obtain that the approximate and correct solutions agree with the exact solutions.

6. CONCLUSIONS

In this paper, an efficient numerical algorithm was presented based on the combination of shifted Chebyshev polynomials and variable-order fractional differential properties. It was applied to solve a set of variable-order fractional partial differential equations. And the relevant theoretical knowledge of error correction was proposed. In fact, we transformed the original equation sets into the products of some matrices, then used Matlab to solve the problem. The errors of approximate results were corrected so that the absolute errors can be reduced. Finally, three numerical examples were presented to illustrate the efficiency of the proposed method. Only small values of

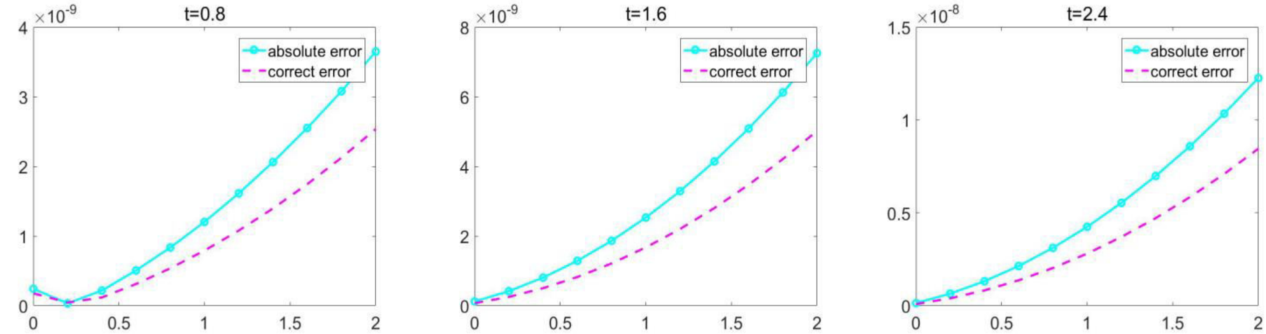

Figure 13: Absolute errors and correct errors of $v(x,t)$ for Example 3, when $n=2$.

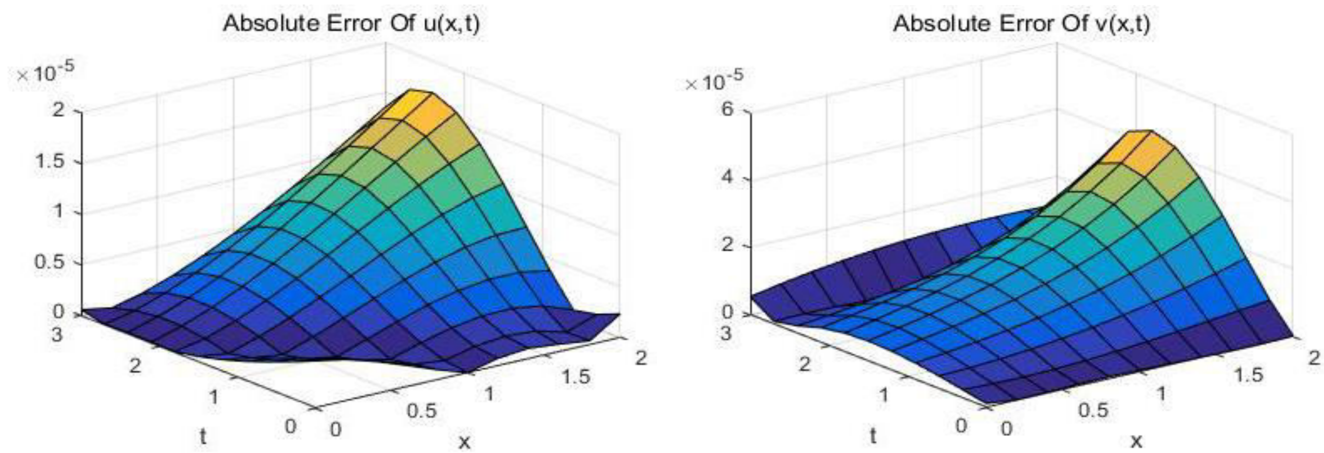

Figure 14: Absolute errors between approximate and exact solutions for Example 3, when $n=3$.

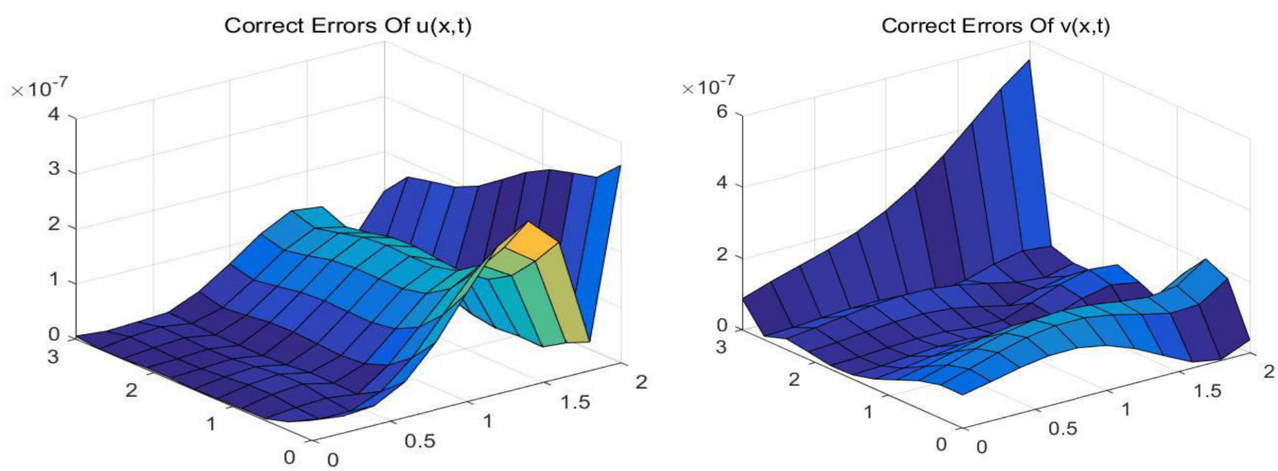

Figure 15: Correct errors between approximate and exact solutions for Example 3, when $n=3$.

n were appropriately selected so that a high convergence precision can be reached to 10^{-9} and 10^{-8}.

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